FINAL REPORT

EARTH SATELLITE ORBIT COMPUTATIONS

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27 AUGUST 1962

PART I
DILIBERTO THEORY

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SPACE TECHNOLOGY LABORATORIES, INC.
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Part I

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27 August 1962

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National Aeronautics and Space Administration
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ABSTRACT

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The Diliberto general perturbation method based on the theory of periodic surfaces has been applied in a new and more suitable coordinate system. As a result, analysis and results have been considerably simplified. An analytic treatment of the time-angle relationship for this coordinate system has been developed. By introducing a further change of variables, low eccentricity singularities have been eliminated. The special cases of polar and equatorial orbits have been examined, and convergence of the method demonstrated in the latter case. Both the basic and the low eccentricity method have been tested numerically. author

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FINAL REPORT

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Part I - The Diliberto Theory

INTRODUCTION

The earth's axisymmetric gravitational potential is conveniently written as a function of two variables u and v, where u is the reciprocal of the geocentric distance and v is the sine of the geocentric latitude. The orbit of a near earth satellite is then essentially determined by the solutions of the non-linear ordinary differential equations

$$\frac{d^2u}{dw^2} + u = F_1 \left(u, \frac{du}{dw}, v, \frac{dv}{dw}, \lambda \right),$$
$$\frac{d^2v}{dw^2} + v = F_2 \left(u, \frac{du}{dw}, v, \frac{dv}{dw}, \lambda \right), \tag{1}$$

where w is a timelike variable, and λ is a small parameter measuring the oblateness of the earth.

It can be shown that $v = \sin i \sin \beta$, $dv/dw = \sin i \cos \beta$, where i is the inclination of the orbit plane and β is the argument of the latitude.

Therefore, the solutions of (1) determine the position of the satellite in the orbit plane together with the plane's inclination. The longitude of the ascending node and the relationship between w and t must be found by quadratures.

Since an exact representation of the solutions is unavailable except in certain special cases, such as an orbit lying in the earth's equatorial plane, approximating formulas have been generated by various methods. Most of these methods assume that the solutions can be represented by expansions in terms of doubly periodic functions u_n, v_n,

$$u = \sum_{n=0}^{\infty} \lambda^n u_n (\phi_1, \phi_2)$$

$$v = \sum_{n=0}^{\infty} \lambda^n v_n (\phi_1, \phi_2) \tag{2}$$

where the variables ϕ_1 and ϕ_2 are solutions to

$$\frac{d\phi_j}{dw} = 1 + \sum_{n=1}^{\infty} \lambda^n \Phi_{nj} (\phi_1, \phi_2). \tag{3}$$

In this context, a doubly periodic function has period 2π in each of the variables ϕ_1 and ϕ_2.

There is considerable flexibility in the selection of the variables ϕ_1 and ϕ_2. Because of this, term by term comparison of different general perturbation schemes is extremely difficult. The variables ϕ_j are usually related to inplane angles such as the true anomaly or the argument of the latitude, but this is not always the case. The differential equations (3) are sometimes defined inductively, using the requirement that the function u_n and v_n be double periodic. If this approach is used, some process of "removing the secular terms" is required.

Equations (2) and (3) define a transformation of coordinates from u, du/dw, v, dv/dw to u, v, ϕ_1, ϕ_2. Unless care is taken, this transformation can be singular for nearly circular orbits. Another feature of general perturbation schemes is the failure of the terms in the expansion (2) to be defined at the critical inclination, $\cos^2 i = 1/5$.

The representation (2) can be interpreted as defining a torus-like surface in the four dimensional phase space of the variables u, du/dw, v, dv/dw. This periodic surface is an invariant manifold of the system (1) in that it is generated by the solutions to the differential equations. The motion on the surface is governed by equation (3).

S. P. Diliberto recognized that the invariant manifold which is implicit in the representation (2) can be considered as the intersection of the manifolds defined by two periodic integrals, the first of these being the known energy integral. The second, he conjectured to be associated with the angular momentum of the satellite in that the vanishing of the oblateness parameter λ implies that the second integral is a statement of the conservation of angular momentum.

This geometric interpretation has led to a systematic method [2,3] for generation the terms in the expansion (2). This method has three principal steps:

1) Developing an expansion for the conjectured periodic integral,

2) Inverting this integral simultaneously with the known energy integral to obtain equations of a surface which approximates the periodic surface,

3) Deriving approximations to the solutions on the surface.

Further details are given in Sections 3 and 4, where some material from previous work [2, 3] is repeated, considerably revised, for the sake of better exposition. It has been shown that a formal expansion of this sort exists; that is, the doubly periodic functions u_n and v_n can be calculated, but the convergence of the process has not been established except for the somewhat trivial case of equatorial orbits (Section 9).

In this report, the Diliberto expansion procedure is applied in a coordinate system which proves to be more suitable than that introduced in [3], as is quite evident when corresponding formulas are compared. Despite this simplification, extension of the results to second order effects of the second harmonic and first order effects of the third and fourth harmonics in the gravitational potential is too large a task to include in the present study. In the new coordinate system, as well as in the old, singularities arise at certain special values of the parameters. Those occurring at zero eccentricity are removed by a further change of variable set forth in Section 2, and polar ($\cos i = 0$) orbits are treated in Section 8, but near-polar ($\cos i \doteq 0$) and near-critical ($\cos^2 i \doteq 1/5$) inclinations are not considered. FORTRAN programs for the IBM 7090 have been written and used in numerical tests of both the basic and the low eccentricity methods; summaries of formulas and results will be found in Sections 5 and 7.

As indicated above, once (1) is solved it is still necessary to find Ω, the longitude of the ascending node, and the relationship between w and t by quadratures. The analysis for Ω is trivial, but the angle-time relationship is another matter. The problem is attacked in Section 6 and a solution is, indeed, obtained; however, one which, on account of its extreme complexity, leaves much to be desired.

Two further aspects of the study may be mentioned: the example of Section 10 which shows how an inopportune choice of coordinates may conceal the existence of periodic surfaces, and the qualitative properties of equatorial orbits deduced in Section 9.

In addition to S. P. Diliberto, contributors to this report are E. B. Collins, A. H. Halpin, W. T. Kyner, B. Sherman, and O. K. Smith. Responsibility for final revision and editing rests with the last named. The two programs were written by J. Ayers, A. H. O'Leary, and J. L. Tobey.

1. EQUATIONS OF MOTION

The differential equations describing the motion of a near earth satellite have been expressed in many coordinate systems. Here they are derived relative to a moving coordinate system which has one axis parallel to the angular momentum vector and another parallel to the position vector. The final form of the differential equations is suitable for application of the Diliberto expansion procedure described in subsequent sections.

The center of the earth is taken as the origin of the coordinate system and is assumed to be fixed in space. The position of the satellite at time t is defined by the vector $\underline{r}(t)$ satisfying the equation

$$\frac{d^2}{dt^2}\underline{r} = \underline{F} \tag{1.1}$$

Since we neglect all forces except the gravitational attraction of the earth, the force is derivable from a potential function U

$$\underline{F} = - \text{grad } U$$

The moving coordinate system is determined by three unit vectors $\underline{P}$, $\underline{Q}$, and $\underline{R}$. The vector $\underline{P}$ is parallel to the position vector $\underline{r}$; the vector $\underline{R}$ is parallel to the angular momentum vector $\underline{H}$. Therefore

$$\underline{H} = \underline{r} \times \underline{v} = H\underline{R},$$

$$\underline{r} = r\underline{P},$$

$$\underline{Q} = \underline{R} \times \underline{P},$$

where H and r are the lengths of the vectors $\underline{H}$ and $\underline{r}$ and $\underline{v} = d\underline{r}/dt$. The moving triad $\underline{P}$, $\underline{Q}$, $\underline{R}$ is described relative to an inertial triad $\underline{i}$, $\underline{j}$, $\underline{k}$ ($\underline{k}$ directed northward along the polar axis) by the Eulerian angles i, Ω, β where

i = the inclination of the orbit plane,

Ω = the longitude of the ascending node

β = the argument of the latitude

(see [6], pages 183 - 184; we use β instead of u).

Coordinates of a vector relative to the two coordinate systems are related by the equations ([4], page 109)

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} F_1' \\ F_2' \\ F_3' \end{bmatrix} \tag{1.2}$$

where the primed components relate to the rotating system and Λ is the matrix with elements

$$a_{11} = \cos\beta \cos\Omega - \cos i \sin\Omega \sin\beta$$

$$a_{12} = \cos\beta \sin\Omega + \cos i \cos\Omega \sin\beta$$

$$a_{13} = \sin\beta \sin i$$

$$a_{21} = -\sin\beta \cos\Omega - \cos i \sin\Omega \cos\beta$$

$$a_{22} = -\sin\beta \sin\Omega + \cos i \cos\Omega \cos\beta$$

$$a_{23} = \cos\beta \sin i$$

$$a_{31} = \sin i \sin\Omega$$

$$a_{32} = -\sin i \cos\Omega$$

$$a_{33} = \cos i$$

Since Λ is a rotation matrix, it has an inverse, Λ^{-1}, equal to its transpose.

If the earth is assumed axisymmetric, the potential has the familiar expansion in Legendre polynomials [1]

$$U = -\mu \sum_{n=0}^{\infty} B_n P_n(z/r) r^{-n-1} \qquad (B_0=1, B_1=0) \tag{1.3}$$

with (x,y,z) the coordinates of a point referred to the inertial system. From the general equations (1.2), we have

$$z = a_{13}x' + a_{23}y' + a_{33}z'$$

and since at the satellite $x' = r$, $y' = z' = 0$ and

$$\operatorname{grad}(z/r) = (a_{23}\,\underline{Q} + a_{33}\underline{R})/r$$

the force (per unit mass) acting on the satellite is given by

$$
\begin{aligned}
\underline{F} &= - \text{grad } U \\
&= \mu(a_{23}\underline{Q} + a_{33}\underline{R}) \sum_{n=2}^{\infty} B_n P_n' (a_{13})\, r^{-n-2} \\
&\qquad - \mu\, \underline{P} \sum_{n=0}^{\infty} (n+1)\, B_n P_n (a_{13})\, r^{-n-2}
\end{aligned}
\tag{1.4}
$$

In order to find the rates of change of the Euler angles, the angular velocity of the rotating system is needed. It is given by the vector

$$\underline{\omega} = (\underline{F}\cdot\underline{R})\underline{r}/H + H\, \underline{R}/r^2 . \tag{1.5}$$

To verify this we note that differentiating $\underline{H} = \underline{r} \times \underline{v}$ gives

$$\frac{d}{dt}\underline{H} = \underline{v} \times \underline{v} + \underline{r} \times \frac{d}{dt}\underline{v} = \underline{r} \times \underline{F} , \tag{1.6}$$

$$\underline{R} \times \frac{d}{dt}\underline{H} = \underline{R} \times (\underline{r} \times \underline{F}) = (\underline{R}\cdot\underline{F})\underline{r} ,$$

while differentiating $\underline{H} = H\underline{R}$ gives

$$\underline{R} \times \frac{d}{dt}\underline{H} = \underline{R} \times \left(\frac{dH}{dt}\underline{R} + H \frac{d}{dt}\underline{R}\right) = H(\underline{R} \times \frac{d}{dt}\underline{R}) .$$

Therefore

$$\underline{R} \times \frac{d}{dt}\underline{R} = (\underline{R}\cdot\underline{F})\underline{r}/H$$

Now resolve $\underline{\omega}$ into components in the rotating system

$$\underline{\omega} = \omega_1 \underline{P} + \omega_2 \underline{Q} + \omega_3 \underline{R} .$$

Since ([4], page 133)

$$\frac{d}{dt}\underline{R} = \underline{\omega} \times \underline{R} \ ,$$

$$\underline{R} \times \frac{d}{dt}\underline{R} = \underline{R} \times (\underline{\omega} \times \underline{R}) = \underline{\omega} - \omega_3 \underline{R} \ .$$

So, by (1.5)

$$\omega_1 \underline{P} + \omega_2 \underline{Q} = \underline{\omega} - \omega_3 \underline{R} = (\underline{R}\cdot\underline{F}) \underline{r}/H$$

Hence, comparing components,

$$\omega_1 = (\underline{R}\cdot\underline{F}) r/H$$

$$\omega_2 = 0 \ . \tag{1.7}$$

To find ω_3, we note that

$$\underline{v} = \frac{d}{dt}(r\underline{P}) = \frac{dr}{dt}\underline{P} + r(\underline{\omega} \times \underline{P}) = \frac{dr}{dt}\underline{P} + r\,\omega_3 \underline{Q} \ . \tag{1.8}$$

Hence the angular momentum vector may be expressed

$$\underline{H} = \underline{r} \times \underline{v} = r^2\omega_3(\underline{P} \times \underline{Q}) \ ,$$

and the angular momentum in the orbital plane (the plane of $\underline{P}$ and $\underline{Q}$) is simply

$$\omega_3 = H/r^2 \ .$$

In order to use the formulas from [4] which give the relationship of the angular velocity and the Eulerian angles, we note that Ω corresponds to ϕ, β to ψ, and i to θ. We have ([4], pages 107 and 134).

$$\omega_1 = \frac{d\Omega}{dt} \sin i \sin \beta + \frac{di}{dt} \cos \beta$$

$$0 = \frac{d\Omega}{dt} \sin i \cos \beta - \frac{di}{dt} \sin \beta$$

$$\omega_3 = \frac{d\Omega}{dt} \cos i + \frac{d\beta}{dt} .$$

Therefore

$$\frac{di}{dt} = \omega_1 \cos \beta$$

$$\frac{d\Omega}{dt} = \omega_1 \sin \beta / \sin i$$

$$\frac{d\beta}{dt} = \omega_3 - \omega_1 \sin \beta \cot i.$$

A new independent variable w is defined by

$$\frac{dw}{dt} = \omega_3 = H/r^2 = p/(r^2 \cos i)$$

where $p = \underline{H} \cdot \underline{k} = H \cos i$, the component of angular momentum along the polar axis.

By definition, $\frac{dw}{dt}$ is the angular rate of change of the position vector in the orbital plane, hence

$$\frac{di}{dw} = (r^2\omega_1/p) \cos i \cos \beta$$

$$\frac{d\Omega}{dw} = (r^2\omega_1/p) \cot i \sin \beta \qquad (1.9)$$

$$\frac{d\beta}{dw} = 1 - (r^2\omega_1/p) \cos i \cot i \sin \beta$$

The polar angular momentum is one of the two constants of the motion. To verify this, differentiate the definition $p = \underline{H}\cdot\underline{k}$ and apply (1.6). There results

$$\frac{dp}{dt} = \frac{d\underline{H}}{dt} \cdot \underline{k} = (\underline{r} \times \underline{F}) \cdot \underline{k}$$

As a consequence of the assumed symmetry, the force $\underline{F} = -$ grad U lies in the plane of $\underline{r}$ and $\underline{k}$. The product $\underline{r} \times \underline{F}$ is accordingly perpendicular to $\underline{k}$ and

$$\frac{dp}{dt} = \frac{dp}{dw} = 0 \qquad (1.10)$$

Further, it follows by differentiating $p = H \cos i$ and utilizing (1.9) that

$$\frac{1}{H} \frac{dH}{dw} = (r^2\omega_1/p) \sin i \cos \beta \qquad (1.11)$$

Now let us introduce as one of our two principal independent variables $u = 1/r$. Then

$$\frac{dw}{dt} = Hu^2$$

$$\frac{dr}{dt} = - \frac{Hdu}{dw} \tag{1.12}$$

$$\frac{d^2r}{dt^2} = - H^2u^2 \frac{d^2u}{dw^2} - H u^2 \frac{dH}{dw} \frac{du}{dw} .$$

Derivation of a differential equation in u begins by differentiating the relationship

$$\underline{r} \cdot \underline{v} = r \frac{dr}{dt}$$

to obtain

$$\underline{v} \cdot \underline{v} + \underline{r} \cdot \frac{d\underline{v}}{dt} = (\frac{dr}{dt})^2 + r \frac{d^2r}{dt^2}$$

Equation (1.8) provides another expression involving $\underline{v} \cdot \underline{v}$:

$$\underline{v} \cdot \underline{v} = (\frac{dr}{dt})^2 + r^2 (\frac{dw}{dt})^2 \tag{1.13}$$

and combining these two yields

$$\underline{r} \cdot \frac{d\underline{v}}{dt} = r \frac{d^2r}{dt^2} - r^2 (\frac{dw}{dt})^2$$

With the introduction of u and w through (1.12), the equation becomes

$$H^2u (\frac{d^2u}{dw^2} + u + \frac{1}{H} \frac{dH}{dw} \frac{du}{dw}) = - \underline{r} \cdot \frac{d\underline{v}}{dt}$$

Replacing the acceleration by the force and making use of (1.11), we finally have the desired result.

$$\frac{d^2u}{dw^2} + u = - (r^2\omega_1/p) \sin i \cos \beta \frac{du}{dw} - (\frac{\cos i}{up})^2 \underline{P} \cdot \underline{F} \tag{1.14}$$

For future reference, we note that the energy (per unit mass) is

$$E = \frac{1}{2} \underline{v} \cdot \underline{v} + U$$

or, by virtue of (1.12) and (1.13)

$$E = \frac{1}{2} H^2 \left[\left(\frac{du}{dw}\right)^2 + u^2 \right] + U \qquad (1.15)$$

By differentiating the first of these with respect to time, it is easy to verify that the energy is, indeed, a second constant of the motion.

Next let us introduce the second of our two independent variables, the sine of the geocentric latitude of the satellite

$$v = \sin i \sin \beta \qquad (1.16)$$

Differentiating and substituting from (1.9) gives

$$\frac{dv}{dw} = \sin i \cos \beta$$

$$\frac{d^2 v}{dw^2} + v = (r^2 \omega_1 / p) \cos^2 i$$

According to (1.7) the quantity in parenthesis is related to the force as follows

$$(r^2 \omega_1 / p) = (\underline{R} \cdot \underline{F}) r^3 \cos i / p^2 = \mu (\cos i / p)^2 \sum_{n=2}^{\infty} B_n P_n'(v) u^{n-1} \qquad (1.17)$$

so after collecting results and making use of expansion (1.4), we have the following system of differential equations

$$\frac{d^2u}{dw^2} + u = - (r^2\omega_1/p)\,\frac{du}{dw}\,\frac{dv}{dw} + \mu\,(\cos\, i/p)^2 \sum_{n=0}^{\infty} (n+1)\, B_n P_n(v)\, u^n$$

$$\frac{d^2v}{dw^2} + v = (r^2\omega_1/p)\,\cos^2 i$$

$$\frac{d\Omega}{dw} = (r^2\omega_1/p)\, v\, \cot\, i\, \csc\, i \tag{1.18}$$

$$\frac{dp}{dw} = 0$$

with $(r^2\omega_1/p)$ expressed by the series above and the trigonometric functions related to the independent variable v by $\sin^2 i = v^2 + (\frac{dv}{dw})^2$. This sixth order system of differential equations in u, v, Ω, and p is equivalent to the vector equation of motion (1.1).

In most of the subsequent analysis, harmonics beyond the second will be neglected. In this case, we have

$$(r^2\omega_1/p) = - 2\lambda\, uv\, \cos^2 i$$

$$= - 2\lambda\, uv\, (1-v^2 - v'^2) \tag{1.19}$$

and, omitting the trivial equation $p' = 0$, the system simplifies to

$$u'' + u = (1-v-v'^2)[A+2\lambda uu'vv'+ \lambda u^2(1-3v^2)]$$

$$v'' + v = - 2\,\lambda uv\,(1-v^2-v'^2)^2 \tag{1.20}$$

$$\Omega' = - 2\,\lambda uv^2(1-v^2-v'^2)^{3/2}/(v^2+v'^2)$$

where $A = \mu/p^2$, $\lambda = - 3AB_2/2$, and primes indicate differentiation with respect to w. Similarly, the expression for the energy (1.15) reduces to

$$E = \frac{\mu}{A}\left[\frac{u^2 + u'^2}{2(1-v^2-v'^2)} - Au - \frac{\lambda}{3}u^3(1-3v^2)\right] \tag{1.21}$$

Note that Ω does not appear on the right hand side of any equation. We may, therefore, set aside the third, solve the first two simultaneously by the Diliberto expansion procedure, and then obtain Ω by quadratures. In the sections which follow, details of this development will be discussed.

2. COORDINATE SYSTEMS

Effective application of the theory of periodic surfaces depends upon choosing a coordinate system in the phase space so that the periodic surfaces have simple representations. (In this connection, see Section 10). Two such coordinate systems, one suitable when the eccentricity is not too small and a second which is free from this restriction, are developed in the present section.

We have from (1.20) the equations

$$\begin{aligned} v'' + v &= \lambda G(v,v')uv \\ u'' + u &= g\left(\sqrt{v^2 + v'^2}\right) + \lambda\, uu'G_1(v,v') + \lambda\, u^2G_2(v,v'), \end{aligned} \tag{2.1}$$

together with the energy integral (1.21)

$$u'^2 + (u-g)^2 = g^2 + 2Eg/\mu + (2\lambda/3A)gu^3(1 - 3v^2) \; . \tag{2.2}$$

Here

$$g(\sin\, i) = A \cos^2 i = \mu/H^2$$

$$G(v,v') = -\, 2(1-v-v'^2)^2$$

$$G_1(v,v') = 2\, vv'(1-v^2-v'^2)$$

$$G_2(v,v') = (1-3v^2)(1-v^2-v'^2)$$

If $\lambda = 0$, then $(v^2 + v'^2)^{1/2}$ (= sin i) is a constant of the motion which is simply related to the magnitude of the angular momentum vector, that is

$$(1 - v^2 - v'^2)H^2 = p^2.$$

Our basic assumption is that there exists an analytic integral

$$\mathcal{H}(v,v',u,u',\lambda) = \text{constant} \tag{2.3}$$

such that

$$\mathcal{H}(v,v',u,u',0) = (v^2 + v'^2)^{1/2} .$$

Such an integral is necessarily independent of the energy integral. Although a mathematical proof of the existence of the angular momentum integral has yet to be given, there are sound reasons for using it as a basis of an expansion procedure. Among these is the role of this integral in the Hansen general perturbation scheme which was used so successfully in the Vanguard program. It can be shown (unpublished result of W. T. Kyner) that the convergence of Hansen's method implies the existence of the integral (2.3).

The pair of integrals, the known energy integral and the conjectured angular momentum integral, define surfaces in the four dimensional phase space of the variables v,v',u,u' whose intersection is homeomorphic to a torus. This torus-like surface, as are the integrals, is generated by solutions to the differential equations (2.1). It is central to our study, giving us both information about the geometry of the phase space and a framework for our approximation scheme.

If λ is equal to zero, the surface is in fact a torus which is most easily represented in polar coordinates

$$v = r_1 \sin \theta_1, \qquad u = g(r_1) + r_2 \sin \theta_2$$
$$v' = r_1 \cos \theta_1, \qquad u' = r_2 \cos \theta_2. \tag{2.4}$$

The equation of the torus is simply

$$r_j = p_j, \quad j = 1,2,$$

where the constants p_j are

$$p_1 = \sin i_o, \text{ the angular momentum integral,}$$

$$p_2 = \left[g^2(r_1) + 2Eg(r_1)/\mu \right]^{1/2}, \text{ the energy integral.}$$

Using this coordinate system with non-zero λ, the intersection of the two independent integrals can be described as a <u>periodic two surface</u>.

DEFINITION: A periodic two surface of equation (2.1) is the graph of a pair of analytic functions

$$r_j = S_j(\theta_1, \theta_2, \lambda) \tag{2.5}$$

defined for all θ_1, θ_2, and for some neighborhood of $\lambda = 0$, with period 2π in θ_j, and such that if

$$p_j = S_j(k_1, k_2, \lambda),$$

then the solution functions $r_j(w)$, $\theta_j(w)$ taking on the initial values

$$r_j(w_o) = p_j, \ \theta_j(w_o) = k_j,$$

satisfy

$$r_j(w) = S_j(\theta_1(w), \theta_2(w), \lambda)$$

for $-\infty < w < +\infty$.

The remainder of this section consists of a discussion of the two coordinate systems which were used and a derivation of the "normal forms" of the differential equations. At a first reading to get the essentials of the method of periodic surfaces, it may be well to skip directly to the treatment of the expansion procedure in the next section.

The coordinate system which has been introduced enables us to define the periodic two surface in terms of classical variables.

For example,

$r_1 = \sin i$, the sine of the inclination of the orbit plane,

$\theta_1 = \beta$, the argument of the latitude,

$\theta_2 + \pi/2 =$ the true anomaly

$r_2/g = e$, the instantaneous eccentricity.

We can therefore see that $r_1 = 0$ implies that the satellite is in the equatorial plane, and $r_2 = 0$ implies that the instantaneous eccentricity is zero.

Angular variables are used extensively in orbit problems, but they can introduce analytic difficulties, e.g. the failure of some general perturbations methods when the eccentricity is small. This is due to the fact that the Jacobian of the transformation (2.4) is equal to the product $r_1 r_2$. Clearly, the circumstances under which either factor vanishes requires study.

Let us first consider the vanishing of r_1. If only even harmonics are present in the potential function (1.3), then orbits in the equatorial plane are possible, i.e. $r_1 \equiv 0$ defines a class of orbits at zero inclination. Furthermore, since $r_1 = 0$ implies that both the velocity and acceleration vectors are parallel to the equatorial plane, r_1 cannot vanish at isolated times. If it is ever zero, then it is identically zero. The differential equations (2.1) become

$$u'' + u = A + \lambda u^2, \qquad v \equiv 0. \tag{2.6}$$

It is easy to see that this equation has periodic solutions which can be represented as elliptic functions. We therefore have the desirable situation that $r_1 = 0$ implies $r_1 \equiv 0$, and the remaining equation has periodic solutions for a wide class of initial conditions.

It is worth noting that the circular orbit in the equatorial plane defined by $r_1 = 0$, $u = A + \lambda u^2$ has constant but non-zero instantaneous eccentricity. Furthermore, other periodic solutions do not in general correspond to closed paths, since the period of the solution need not be commensurable with 2π.

Not let us consider $r_2 = 0$, that is, an orbit with zero eccentricity. Since $r_2 = 0$ implies that $u = g(r_1)$, $u' = 0$, it follows from (2.1) that this can only happen at isolated times. At such a time, θ_2 is not well defined. It is not surprising that many approximation schemes using classical angular variables give poor results for orbits with small eccentricity.

In order to introduce a coordinate system having the property that $r_2 = 0$ implies $r_2 \equiv 0$, and that the remaining differential equation has periodic solutions, we let

$$\begin{aligned} z_1 &= u - g - \sum_{k=1}^{\infty} \lambda^k \alpha_{1k}(v, v') , \qquad z_1 = r_2 \sin \theta_2 , \\ z_2 &= u' - \sum_{k=1}^{\infty} \lambda^k \alpha_{2k}(v, v') , \qquad z_2 = r_2 \cos \theta_2 , \end{aligned} \tag{2.7}$$

where the α_{jk} are to be determined so that the new polar coordinates have the desired property. In other words, if

$$\frac{dz_1}{dw} = z_2 + \sum_{k=1}^{\infty} \lambda^k Z_{1k}(v,v',z_1,z_2)$$

$$\frac{dz_2}{dw} = - z_1 + \sum_{k=1}^{\infty} \lambda^k Z_{2k}(v,v',z_1,z_2), \tag{2.8}$$

then the α_{jk} should be selected so that

$$z_{jk}(v,v',0,0) = 0 \ . \tag{2.9}$$

Furthermore, the remaining equation

$$v'' + v = \lambda\, G(v,v')v \left[z_1 + g + \sum_{k=1}^{\infty} \lambda^k \alpha_{1k}(v,v') \right] \tag{2.10}$$

should have periodic solutions for arbitrary initial inclination.

The determination of such a transformation is equivalent to finding periodic solutions of the differential equations (2.1) having the property that as λ tends to zero, these solutions become the circular orbits at the prescribed inclinations. The mathematical validity of this procedure has not yet been established, except in the special case of equatorial orbits. However the first and second order α_{jk} terms have been calculated, and to that order, (2.9) is satisfied. The corresponding approximate solutions to (2.10) have been obtained as periodic functions of w.

The convergence of the transformation (2.7) would imply that $r_2 = 0$ only if $r_2 \equiv 0$ and that (2.10) has periodic solutions for arbitrary initial inclination. The second statement follows from the observation that if $r_2 \equiv 0$, the energy integral (2.2) determines simple closed curves in the v, v' plane which correspond to periodic solutions of the differential equation (2.10).

In order to calculate the α_{jk}, it is convenient to replace v and v' by the polar coordinates r_1, θ_1 where, as before, $v = r_1 \sin \theta_1$, $v' = r_1 \cos \theta_1$. Recalling that $r_1 = \sin i$, $\theta_1 = \beta$, we have from (1.20),

$$\begin{aligned} \frac{d\theta_1}{dw} &= 1 + 2\lambda u (1 - r_1^2)^2 \sin^2 \theta_1, \\ \frac{dr_1}{dw} &= -\lambda u (1 - r_1^2)^2 r_1 \sin 2\theta_1, \end{aligned} \tag{2.11}$$

where, using (2.7),

$$u = z_1 + g(r_1) + \sum_{k=1}^{\infty} \lambda^k \alpha_{1k} (r_1, \theta_1)$$

If we now differentiate the expansions (2.7) of z_1 and z_2 and replace d^2u/dw^2, dr_1/dw, $d\theta_1/dw$ by their equivalent expressions from the differential equations (2.1) and (2.11) we obtain equations

(2.8), where

$$z_{11} = -\frac{\partial\alpha_{11}}{\partial\theta_1} + \alpha_{21} - 2A(1-r_1^2)^2\ r_1^2\ [z_1 + A(1-r_1^2)]\ \sin 2\ \theta_1$$

$$z_{12} = (1-r_1^2)^2\ [z_1 + A(1-r_1^2)]\ \left(\frac{\partial\alpha_{11}}{\partial r_1} r_1 \sin 2\ \theta_1 - 2\frac{\partial\alpha_{11}}{\partial\theta_1}\sin^2\theta_1\right)$$

$$-\frac{\partial\alpha_{12}}{\partial\theta_1} + \alpha_{22} - 2\alpha_{11}\ A(1-r_1^2)^2\ r_1^2 \sin 2\ \theta_1 \tag{2.12}$$

$$z_{21} = -\frac{\partial\alpha_{21}}{\partial\theta_1} - \alpha_{11} + (1-r_1^2)\ [z_1 + A(1-r_1^2)]\{z_2 r_1^2 \sin 2\theta_1$$

$$+ (1-3r_1^2 \sin^2\theta_1)[z_1 + A(1-r_1^2)]\}$$

$$z_{22} = (1-r_1^2)^2[z_1 + A(1-r_1^2)]\left(\frac{\partial\alpha_{21}}{\partial r_1} r_1 \sin 2\theta_1 - 2\frac{\partial\alpha_{21}}{\partial\theta_1}\sin^2\theta_1\right)$$

$$-\frac{\partial\alpha_{22}}{\partial\theta_1} - \alpha_{12} + (1-r_1^2)\ r_1^2 \sin 2\ \theta_1\ \{z_2\ \alpha_{11} + [z_1 + A(1-r_1^2)]\alpha_{21}$$

$$+ 2\alpha_{11}\ (1-3r_1^2 \sin^2\theta_1)(1-r_1^2)[z_1 + A(1-r_1^2)]$$

The requirement that $z_1 = z_2 = 0$ be a solution means that (2.9) must be satisfied, that is

$$\frac{\partial \alpha_{11}}{\partial \theta_1} - \alpha_{21} + 2A^2(1-r_1^2)^3\, r_1^2 \sin 2\theta_1 = 0$$

$$A(1-r_1^2)^3 \left(\frac{\partial \alpha_{11}}{\partial r_1} r_1 \sin 2\theta_1 - 2 \frac{\partial \alpha_{11}}{\partial \theta_1} \sin^2 \theta_1 \right)$$

$$- \frac{\partial \alpha_{12}}{\partial \theta_1} + \alpha_{22} - 2\alpha_{11}\, A(1-r_1^2)^2 r_1^2 \sin 2\, \theta_1 = 0$$

$$(2.13)$$

$$\frac{\partial \alpha_{21}}{\partial \theta_1} + \alpha_{11} - A^2(1-r_1^2)^3(1-3r_1^2 \sin^2 \theta_1) = 0$$

$$A(1-r_1^2)^3 \left(\frac{\partial \alpha_{21}}{\partial r_1} r_1 \sin 2\, \theta_1 - 2 \frac{\partial \alpha_{21}}{\partial \theta_1} \sin^2 \theta_1 \right) - \frac{\partial \alpha_{22}}{\partial \theta_1} - \alpha_{12}$$

$$+ A(1-r_1^2)^2[\alpha_{21}\, r_1^2 \sin 2\theta_1 + 2\, \alpha_{11}(1-3r_1^2 \sin^2 \theta_1)] = 0$$

The first and third inhomogeneous differential equations determine α_{11} and α_{21} as periodic functions of θ_1 with r_1 appearing as a parameter. We have

$$\alpha_{11}(r_1,\theta_1) = \gamma_1 \sin \theta_1 + \gamma_2 \cos \theta_1 + A^2(1-r_1^2)^3(1- \tfrac{3}{2} r_1^2 + \tfrac{5}{6} r_1^2 \cos 2\, \theta_1)$$

$$(2.14)$$

$$\alpha_{21}(r_1,\theta_1) = \gamma_1 \cos \theta_1 - \gamma_2 \sin \theta_1 + \tfrac{1}{3} A^2(1-r_1^2)^3\, r_1^2 \sin 2\, \theta_1$$

where γ_1 and γ_2 are determined so that the equations for α_{12} and α_{22} have periodic solutions. It is easy to verify that $\gamma_1 = \gamma_2 = 0$. The solutions to the other two are

$$\alpha_{12}(r_1,\theta_1) = A^3(1-r_1^2)^5[(2-\frac{16}{3}r_1^2 + \frac{17}{4}r_1^4) + \frac{1}{9}r_1^2(44-61r_1^2)\cos 2\theta_1 + \frac{3}{4}r_1^4 \cos 4\theta_1] \tag{2.15}$$

$$\alpha_{22}(r_2,\theta_2) = -\frac{1}{9}A^3(1-r_1^2)^5 r_1^2[2(2-r_1^2)\sin 2\theta_1 - 3r_1^2 \sin 4\theta_1]$$

where, once again, the coefficients of $\sin\theta_1$ and $\cos\theta_1$ have been set equal to zero.

We now have

$$\begin{aligned} u &\doteq z_1 + g(r_1) + \lambda\alpha_{11}(r_1,\theta_1) + \lambda^2\alpha_{12}(r_1,\theta_1) \\ \frac{du}{dw} &\doteq z_2 + \lambda\alpha_{21}(r_1,\theta_1) + \lambda^2\alpha_{22}(r_1,\theta_1) \\ v &= r_1 \sin\theta_1 \\ \frac{dv}{dw} &= r_1\cos\theta_1 \ . \end{aligned} \tag{2.16}$$

Before proceeding with the expansion procedure, let us inspect the special case of the equatorial orbit. If the inclination is zero, we have

$$u \doteq z_1 + A + \lambda A^2 + 2\lambda^2 A^3 \ , \tag{2.17}$$

$$\frac{du}{dw} \doteq z_2, \qquad v = 0, \qquad \frac{dv}{dw} = 0 \ .$$

The radius of the circular orbit in the equatorial plane is determined by the quadratic equation obtained from (2.6) by setting $d^2u/dw^2 = 0$. If its radius is denoted by $1/A^*$, then

$$A^*(\lambda) = \frac{1 - (1-4A)^{1/2}}{2} = A + \lambda A^2 + 2\lambda^2 A^3 + \dots \tag{2.18}$$

We now check that the transformation

$$u = z_1 + A^*(\lambda), \qquad \frac{du}{dw} = z_2 \tag{2.19}$$

gives us

$$\frac{dz_1}{dw} = z_2, \quad \frac{dz_2}{dw} = - z_1 + (A - A^*) + \lambda(z_1+A^*)^2 . \tag{2.20}$$

By construction, these equations have $z_1 = z_2 = 0$ as a solution. In other words, the expansion (2.7), when specialized to the case of zero inclination, converges and, furthermore, the periodic solution described by the vanishing of z_1 and z_2 is the circular orbit in the equatorial plane. This orbit obviously has the property of remaining circular as λ tends to zero.

Two moving coordinate systems have now been introduced. The first, which uses classical variables, (see page 21) is given by

$$\begin{aligned} v &= r_1 \sin \theta_1, \quad \frac{dv}{dw} = r_1 \cos \theta_1, \\ u &= r_2 \sin \theta_2 + g(r_1), \quad \frac{du}{dw} = r_2 \cos \theta_2 . \end{aligned} \tag{2.21}$$

The second, even though complicated, is suitable for nearly circular orbits. It is given by

$$
\begin{aligned}
v &= s_1 \sin \phi_1, \quad \frac{dv}{dw} = s_1 \cos \phi_1, \\
u &= s_2 \sin \phi_2 + g(s_1) + \sum_{k=1}^{\infty} \lambda^k \alpha_{1k}(s_1, \phi_1), \\
\frac{du}{dw} &= s_2 \cos \phi_2 + \sum^{\infty} \lambda^k \alpha_{2k}(s_1, \phi_1)
\end{aligned}
\tag{2.22}
$$

(to avoid confusion, we will use s, ϕ as polar variables in the second set).

Relative to the first set of polar variables, the differential equations (2.1) become

$$
\begin{aligned}
\frac{d\theta_1}{dw} &= 1 + 2\lambda u(1-r_1^2)^2 \sin^2 \theta_1 \\
\frac{dr_1}{dw} &= -\lambda u r_1 (1-r_1^2)^2 \sin 2\theta_1 \\
\frac{d\theta_2}{dw} &= 1 - \lambda u(F_1 \cos \theta_2 + F_2 \sin \theta_2)/r_2 \\
\frac{dr_2}{dw} &= -\lambda u(F_1 \sin \theta_2 - F_2 \cos \theta_2)
\end{aligned}
\tag{2.23}
$$

where, for brevity,

$$
\begin{aligned}
u &= A(1-r_1^2) + r_2 \sin \theta_2 \\
F_1 &= 2Ar_1^2(1-r_1^2)^2 \sin 2\theta_1 \\
F_2 &= (1-r_1^2)[r_1^2\, r_2 \sin 2\theta_1 \cos \theta_2 + (1-3r_1^2 \sin^2 \theta_1)u]
\end{aligned}
$$

The presence of r_2 in the denominator of the θ_2 equation serves as an explicit warning that low eccentricity orbits require special care.

In the modified variables, we have

$$\frac{d\phi_1}{dw} = 1 + 2\lambda u(1-s_1^2)^2 \sin^2 \phi_1$$

$$\frac{ds_1}{dw} = - \lambda u s_1(1-s_1^2)^2 \sin 2\phi_1$$

$$\frac{d\phi_2}{dw} = 1 + \frac{1}{s_2}\sum_{k=1}^{\infty}\lambda^k(Z_{1k} \cos \phi_2 - Z_{2k} \sin \phi_2) \tag{2.24}$$

$$\frac{ds_2}{dw} = \frac{1}{s_2} \sum_{k=1}^{\infty} \lambda^k(Z_{1k} \sin \phi_2 + Z_{2k} \cos \phi_2)$$

where we leave

$$u = s_2 \sin \phi_2 + A(1-s_1^2) + \sum_{k=1}^{\infty} \lambda^k \alpha_{1k}(s_1,\phi_1)$$

Clearly $s_2 = 0$ is no longer a cause of concern since the change of variable has been made in such a way that every $Z_{jk}(s_1,s_2,\phi_1,\phi_2)$ is divisible by s_2.

Both sets of equations are in what will be called <u>normal form</u>. They can be studied with the aid of the Diliberto expansion procedure outlined in the introduction and described more fully in the next two sections.

3. THE PERIODIC INTEGRAL

In this section we present an expansion technique for determining an analytic integral

$$\mathcal{H}(\theta_1,\theta_2,r_1,r_2,\lambda) = \text{constant}, \tag{3.1}$$

which has the property that it specializes to the angular momentum integral if the oblateness parameter is set equal to zero. The requirement that $\mathcal{H}$ be doubly periodic with period 2π in θ_1 and θ_2 arises from the change from cartesian to polar coordinates.

The first set of polar coordinates and equations (2.23) will be used in this section since they are somewhat simpler than the second set (2.24). We write

$$\begin{aligned} \frac{d\theta_j}{dw} &= 1 + \lambda\ \Theta_j(\theta_1,\theta_2,r_1,r_2) \\ \frac{dr_j}{dw} &= \lambda R_j(\theta_1,\theta_2,r_1,r_2), \qquad j = 1,2 \end{aligned} \tag{3.2}$$

where the functions Θ_j and R_j are given in (2.23).

We seek an integral having an expansion

$$\mathcal{H} = r_1 + \sum_{n=1}^{\infty} \lambda^n H_n(\theta_1,\theta_2,r_1,r_2) \tag{3.3}$$

If we differentiate (3.1),with respect to w,use the expansion (3.3) and the differential equations (3.2), we obtain the following infinite set of partial differential equations:

$$\frac{\partial H_1}{\partial \theta_1} + \frac{\partial H_1}{\partial \theta_2} = - R_1,$$
$$\frac{\partial H_n}{\partial \theta_1} + \frac{\partial H_n}{\partial \theta_2} = P\, H_{n-1}, \qquad n \geqq 2, \tag{3.4}$$

where

$$P\, H = - \sum_{j=1}^{2} \left[\Theta_j \frac{\partial H}{\partial \theta_j} + R_j \frac{\partial H}{\partial r_j} \right] .$$

If these equations have periodic solutions, then the sequence of functions $\{H_n\}$ can be used to define a formal analytic expansion of the integral $\mathcal{H}$. At this time, no information is available concerning the convergence of the series. As will be shown, an infinite set of conditions must be satisfied so that the functions $\{H_n\}$ will be periodic. These conditions restrict the class of coordinate systems which can be used if a normalized integral (i.e., $\mathcal{H} = r_1$ if $\lambda = 0$) is to exist. For the coordinate systems used in this report it can be shown (unpublished theorem of S. P. Diliberto and W. R. Haseltine) that these conditions are satisfied.

The following elementary theorem is needed for the study of the equations (3.4).

<u>THEOREM</u>: A necessary and sufficient condition that there exist a doubly periodic solution of the differential equation

$$\frac{\partial f}{\partial \theta_1} + \frac{\partial f}{\partial \theta_2} = g(\theta_1, \theta_2) \tag{3.5}$$

is that the doubly periodic function g satisfy the equation

$$Mg \equiv \frac{1}{2\pi}\int_0^{2\pi} g(\theta_1 + t, \theta_2 + t)\, dt = 0 . \tag{3.6}$$

If this condition is satisfied, the general periodic solution is given by

$$f = Qg + h(\theta_2 - \theta_1) \tag{3.7}$$

where h is an arbitrary differentiable periodic function and the operator Q is defined by

$$Qg = \frac{1}{2\pi}\int_0^{2\pi} tg(\theta_1 + t, \theta_2 + t)\, dt. \tag{3.8}$$

It can be verified that $MR_1 = 0$; therefore we can write

$$H_1 = -QR_1 + h_1(\psi), \qquad \psi = \theta_2 - \theta_1. \tag{3.9}$$

The periodic function h_1 is to be selected so that H_2 can be calculated. We therefore require that

$$MP(-QR_1 + h_1) = 0. \tag{3.10}$$

Using the definition of the operator P and the fact that a function of ψ is unaffected by the operator M, we get

$$\frac{dh_1}{d\psi} = - MPQ\ R_1/\Delta\ , \tag{3.11}$$

where

$$\Delta = M(\Theta_2 - \Theta_1) = \frac{A}{2}(1-r_1^2)^2(5\ r_1^2 - 4) \tag{3.12}$$

Since Δ does not vanish except for orbits at the critical inclination, $\cos^2 i = 1/5$, we can satisfy condition (3.10). As stated in the introduction we have not yet made a study of this exception. The satisfaction of (3.11) is not sufficient for the computability of the function H_2, for we must know that $h_1(\psi)$ is periodic. The equation (3.11) will define a periodic function if and only if

$$\int_0^{2\pi} MPQR_1\ d\psi = 0. \tag{3.13}$$

If this condition is satisfied, we can compute H_2. The subsequent steps follow the same pattern. At each step we take

$$H_n = QP\ H_{n-1} + h_n\ (\psi) \tag{3.14}$$

with

$$\frac{dh_n}{d\psi} = MPQP\ H_{n-1}/\Delta\ , \tag{3.15}$$

together with the periodicity condition

$$\int_0^{2\pi} MPQP\ H_{n-1}\ d\psi = 0. \tag{3.16}$$

The infinite set of conditions (3.16) place a restriction on the coordinate system and, ultimately, on the differential equations being studied. As stated, it can be shown that they are satisfied here, and therefore the sequence of doubly periodic function $\{H_n\}$ exists.

Even in the absence of information about the convergence of the series (3.3), the formulas can be used to define torus-like surfaces, which would approximate the conjectured periodic surface. A statement about the degree of the approximation requires, at the very least, knowledge that (3.3) is an asymptotic series.

If we set

$$p_1 = r_1 + \sum_{n=1}^{N} \lambda^n H_n \tag{3.17}$$

and write the known energy integral as

$$r_2^2 = g^2(r_1) + 2Eg(r_1)/\mu + (2\lambda/3A)g(r_1)(1-3\, r_1^2 \sin^2 \theta_1)(r_2 \sin \theta_2 + g)^3 \tag{3.18}$$

then these two equations can be solved by iteration to give

$$r_j = p_j + \sum_{n=1}^{N} \lambda^n S_{jn}(\theta_1, \theta_2) + 0(\lambda^{N+1}) \qquad j = 1,2. \tag{3.19}$$

In this way, we have obtained the equations of our approximating surfaces.

It should be noted that because the functions appearing in the differential equations (2.23) have finite Fourier series in θ_1 and θ_2, this will be true of the functions H_n, h_n and S_{jn}.

The equations (3.19) define a surface which we hope to be close to a surface to which the solutions are constrained. They can be used together with the differential equations for θ_1 and θ_2 to approximate the motion on the surface. This is the subject of the next section.

4. APPROXIMATIONS ON THE SURFACE

We will now consider the problem of approximating the solution of a differential equation defined on a torus-like surface. This equation is obtained by taking the θ_j equations of (3.2) and substituting the periodic surface approximations (3.19). We have

$$\frac{d\theta_j}{dw} = 1 + \lambda\, \Theta_j\Big(\theta_1, \theta_2, r_1(\theta_1,\theta_2,\lambda), r_2(\theta_1,\theta_2,\lambda)\Big), \qquad j = 1,2. \tag{4.1}$$

These equations define a flow on the surface. Since w does not appear explicitly, we can take either θ_1 or θ_2 as a new independent variable. For example, if θ_2 is chosen, then

$$\begin{aligned} \frac{d\theta_1}{d\theta_2} = \frac{1 + \lambda\Theta_1}{1 + \lambda\Theta_2} &\equiv 1 + \lambda\Theta(\theta_1,\ \theta_2,\lambda). \\ &\equiv 1 + \sum_{n=1}^{\infty} \lambda^n\, \Gamma_n(\theta_1,\theta_2) \end{aligned} \tag{4.2}$$

The solution to this equation is an integral curve on the surface. The position of the satellite cannot be found until θ_2 as a function of t is known. The first order solution of the problem of finding this angle-time relationship is given in section 6.

It should be noted that although we are using the notation of one of the two coordinates systems employed in this report, the formalism of this section applies to both.

It is a classical result that the average slope, or more precisely, the rotation number, of a solution to a differential equation such as (4.2) exists and is independent of the initial conditions. Therefore a continuous function $c(\lambda)$ which vanishes at $\lambda = 0$ is defined by

$$\lim_{\theta_2 \to \infty} \theta_1/\theta_2 = 1 + c(\lambda) \; ; \tag{4.3}$$

It is not always true that $c(\lambda)$ is analytic. However it will be shown in this section that if Θ is an analytic function of the three variables θ_1, θ_2 and λ, and if $M\Theta \neq 0$, then $c(\lambda)$ has a formal power series expansion. Furthermore, a change of variables can be made

$$\begin{aligned} q_1 &= \theta_1 + \lambda B(\theta_1, \theta_2, \lambda) \\ q_2 &= \theta_2, \end{aligned} \tag{4.4}$$

so that

$$\frac{dq_1}{dq_2} = 1 + c(\lambda) = 1 + \sum_{n=1}^{\infty} \lambda^n c_n . \tag{4.5}$$

The doubly periodic function $B(\theta_1, \theta_2, \lambda)$ will be given by a formal power series in λ.

Clearly

$$\begin{aligned} \theta_1 &= q_1 - \lambda\, B(q_1, q_2, 0) \\ \theta_2 &= q_2, \\ q_1 &= q_1^0 + (1 + \lambda c_1) q_2 \end{aligned} \tag{4.6}$$

is a formal first order solution of (4.2)

The formal series expansion of B is written

$$\lambda B(\theta_1,\theta_2,\lambda) = \sum_{n=1}^{\infty}\lambda^n[B_n(\theta_1,\theta_2) + b_n(\psi)] \ , \tag{4.7}$$

where $MB_n = 0$, $\psi = \theta_2 - \theta_1$, b_n has period 2π in ψ, and $b_n(0) = 0$. This last condition is an arbitrary determination of a constant of integration; another choice would give a transformation of the same type.

To determine the functions B_n and b_n, we differentiate equation (4.4),and substitute from (4.2), (4.5) and (4.7). After equating coefficients of powers of λ, we obtain an infinite set of equations

$$\frac{\partial B_1}{\partial\theta_1} + \frac{\partial B_1}{\partial\theta_2} = c_1 - \Gamma_1$$

$$\frac{\partial B_n}{\partial\theta_1} + \frac{\partial B_n}{\partial\theta_2} = c_n - \Gamma_n - \frac{db_{n-1}}{d\psi}\,\Gamma_1 - \Gamma_1\frac{\partial B_{n-1}}{\partial\theta_1} - \sum_{k=1}^{n-2}\Gamma_{n-k}\left[\frac{db_k}{d\psi} + \frac{\partial B_{n-k}}{\partial\theta_1}\right] , \qquad n \geq 2. \tag{4.8}$$

As before, the right side of these equations must have zero mean. The periodic functions $b_n(\psi)$ and the constants c_n will be selected inductively so that this condition is satisfied.

From the first equation, we see that $c_1 = M\,\Gamma_1$ (a constant for both coordinate systems). We now use an inductive argument. Since b_n is a function of ψ only, $M\left[\Gamma_1\, db_n/d\psi\right] = c_1\, db_n/d\psi$. Therefore

$$c_1 \frac{db_{n-1}}{d\psi} = c_n - M\left\{\Gamma_n + \Gamma_1 \frac{\partial B_{n-1}}{\partial \theta_1} + \sum_{k=1}^{n-2} \Gamma_{n-k}\left[\frac{db_k}{d\psi} + \frac{\partial B_{n-k}}{\partial \theta_1}\right]\right\} \tag{4.9}$$

This equation contains c_n as a parameter. It is selected by the requirement that b_{n-1} have period 2π in ψ . Hence

$$c_n = \int_0^{2\pi} M\left\{\Gamma_n + \Gamma_1 \frac{\partial B_{n-1}}{\partial \theta_1} + \sum_{k=1}^{n-2} \Gamma_{n-k}\left[\frac{db_k}{d\psi} + \frac{\partial B_{n-k}}{\partial \theta_1}\right]\right\} d\psi \tag{4.10}$$

Having determined c_n and b_{n-1}, the doubly periodic function B_n is found by applying the Q operator to the right side of equation (4.8). For example,

$$B_1 = Q\,[M\,\Gamma_1 - \Gamma_1]$$

$$c_1 b_1 = c_2\psi - \int_0^{\psi} M\left[\Gamma_2 + \Gamma_1 \frac{\partial B_1}{\partial \theta_1}\right] d\psi \tag{4.11}$$

$$c_2 = \int_0^{2\pi} M\left[\Gamma_2 + \Gamma_1 \frac{\partial B_1}{\partial \theta_1}\right] d\psi$$

5. FORMULAS AND NUMERICAL RESULTS-FIRST METHOD

A FORTRAN program, based on the expansion developed in preceeding sections, has been written to calculate the position and velocity components of a satellite. Formulas used are collected here beginning, after a few more preliminaries, in 5.4.

5.1 Notation differs somewhat from that of previous sections. We have

$$\theta_1 = \beta,\ r_2 = \rho,\ \theta_2 = \sigma$$

and the inclination i (rather than sin i) corresponds to r_1, reflecting the fact that programming began before the merits of the present choice were appreciated. With these changes, the system (2.23) becomes

$$\beta' = 1 + 2\,\lambda u \cos^4 i \sin^2 \beta$$

$$\sigma' = 1 - \lambda u(F_1 \cos\sigma + F_2 \sin\sigma)/\rho$$

$$i' = -\,\lambda u \sin i \cos^3 i \sin 2\beta$$

$$\rho' = -\,\lambda u(F_1 \sin\sigma - F_2 \cos\sigma)$$

where, in the interest of brevity,

$$u = \rho \sin\sigma + A\cos^2 i$$

$$F_1 = 2A \sin^2 i \cos^4 i \sin 2\beta$$

$$F_2 = \cos^2 i\,[\rho \sin^2 i \sin 2\beta \cos\sigma + (1-3\sin^2 i \sin^2\beta)u]$$

If, in accordance with Section 4, we adopt σ as independent variable, there results

$$\frac{di}{d\sigma} = \lambda R_1$$

$$\frac{d\rho}{d\sigma} = \lambda R_2$$

$$\frac{d\beta}{d\sigma} = 1 + \lambda \Theta$$

where, to order zero in λ - as is appropriate if the system is to be solved to first order only,

$$\Theta = \Theta_1 - \Theta_2 = \frac{1}{2} \rho \cos^4 i \, [2 \sin (\psi+\beta) - \sin (\psi+3\beta) - \sin(\psi-\beta)]$$

$$+ A \cos^6 i \, (1-\cos 2\beta)$$

$$- \frac{3}{2} A \cos^4 i \sin^2 i \, [\cos(2\psi+4\beta) + 1 - \cos(2\psi+2\beta) - \cos 2\beta]$$

$$+ \frac{1}{4} \frac{A^2}{\rho} \cos^6 i \sin^2 i [7 \sin(3\beta+\psi) + \sin(\beta-\psi) - 6 \sin(\psi+\beta)]$$

$$- \frac{1}{16} \rho \cos^2 i \sin^2 i \, [\sin(3\psi+\beta) + 7 \sin (\beta-\psi)$$

$$+ 5 \sin(5\beta+3\psi) - 11 \sin(\psi+3\beta)$$

$$+ 18 \sin(\psi+\beta) - 6 \sin(3\psi+3\beta)]$$

$$+ \frac{1}{4} \rho \cos^2 i [3 \sin(\psi+\beta) - \sin(3\psi+3\beta)]$$

$$+ A \cos^4 i [1-\cos(2\psi+2\beta)] + \frac{A^2}{\rho} \cos^6 i \sin(\psi+\beta)$$

$$R_1 = -\frac{1}{2}\rho\cos^3 i\sin i[\cos(\beta-\psi) - \cos(3\beta+\psi)]$$

$$-A\cos^5 i\sin i\sin 2\beta$$

$$R_2 = \frac{1}{2}\rho A\cos^4 i\sin^2 i[\sin 2\beta - 3\sin(4\beta+2\psi) + 3\sin(2\beta+2\psi)]$$

$$-\frac{1}{4}A^2\cos^6 i\sin^2 i[\cos(\beta-\psi) - 7\cos(3\beta+\psi) + 6\cos(\psi+\beta)]$$

$$+\frac{1}{16}\rho^2\cos^2 i\sin^2 i[5\cos(\beta-\psi) + \cos(3\beta+\psi)$$

$$-5\cos(5\beta+3\psi) - \cos(3\psi+\beta) - 6\cos(\psi+\beta) + 6\cos(3\psi+3\beta)]$$

$$+A^2\cos^6 i\cos(\psi+\beta)$$

$$+\rho A\cos^4 i\sin(2\beta+2\psi)$$

$$+\frac{1}{4}\rho^2\cos^2 i[\cos(\psi+\beta) - \cos(3\psi+3\beta)]$$

Here, as elsewhere, $\psi = \theta_2 - \theta_1 = \sigma - \beta$. To make it easier to apply the operators M and Q which were defined in Section 3, we have first reduced the functions R_1, R_2, and Θ to a form free from products of trigonometric functions of σ and β and have then eliminated σ.

5.2 A formula for longitude of the node is needed, but this presents no difficulties. From (1.20), after introduction of our present variables, follows

$$\Omega' = -2\lambda u\cos^3 i\sin^2\beta$$
$$= -2\lambda(\rho\sin\sigma + A\cos^2 i)\cos^3 i\sin^2\beta$$

Since, from the equations of the preceeding paragraph it is evident that

$$\sigma' = 1 + O(\lambda)$$

$$i = i_o + O(\lambda)$$

$$\rho = \rho_o + O(\lambda)$$

$$\beta = \beta_o + \sigma - \sigma_o + O(\lambda)$$

with subscript zero denoting values at the initial time, we have to first order in λ

$$\frac{d\Omega}{d\sigma} = - 2\lambda \cos^3 i_o[\rho_o \sin \sigma + A \cos^2 i_o] \sin^2(\sigma-\sigma_o+\beta_o)$$

This differential equation can be integrated without difficulty to yield the result found below in 5.12.

5.3 With the new variables, the energy integral (2.2) becomes

$$\rho^2 = \cos^2 i[A^2 \cos^2 i + 2 E p^{-2} + 2\lambda u^3(\frac{1}{3} - \sin^2 i \sin^2\beta)]$$

In order better to separate terms of different order, $\cos^2 i$ is split.

$$\cos^2 i = \cos^2 i_o + (\cos^2 i - \cos^2 i_o)$$

$$= \cos^2 i_o - \sin (i+i_o) \sin (i-i_o)$$

Introducing a constant

$$\bar{E} = 2Ep^{-2} + A^2 \cos^2 i_o$$

the energy integral becomes

$$\rho^2 = \cos^2 i[\bar{\bar{E}} - \sin(i+i_o) \sin (i-i_o) + 2\lambda u^3(\tfrac{1}{3} - \sin^2 i \sin^2 \beta)]$$

$\bar{\bar{E}}$ may also be evaluated using the energy integral at initial time, from which

$$\bar{\bar{E}} = \frac{\rho_o^2}{\cos^2 i_o} - 2\lambda u_o^3(\tfrac{1}{3} - \sin^2 i_o \sin^2 \beta_o)$$

5.4 The calculations that are performed will now be outlined. Since the angle-time relationship (Section 6) is not included in the present program, it is necessary at each step to compute σ using position and velocity components obtained by some other method, which also provides a standard of comparison for the results. The formulas may be compared with those in [3], in which a different set of coordinates was used.

It will be noted that the new choice of variables greatly simplifies the formulas. Input values are the constants R (the mean radius of the earth), μ, and $J = - 3B_2/2R^2$ together with the initial time t_o, and the initial position and velocity components X_o, Y_o, Z_o, $\dot{X}_o$, $\dot{Y}_o$, $\dot{Z}_o$. The program first computes additional constants of the orbit.

$$p = X_o \dot{Y}_o - \dot{X}_o Y_o , \quad A = \mu/p^2 , \quad \lambda = JAR^2$$

5.5 From initial components (and later from components computed by the comparison method) the program evaluates various variables as follows

$$r^2 = X^2 + Y^2 + Z^2 , \qquad v^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2$$

$$u = 1/r$$

$$W = (W_x, W_y, W_z) = \frac{\underline{r} \times \underline{v}}{|r x v|}$$

$$\sin i = \sqrt{W_x^2 + W_y^2} \quad ; \quad \cos i = W_z \; ; \quad i = \arctan(\sin i/\cos i)$$

$$\rho \sin \sigma = u - A \cos^2 i$$

$$\rho \cos \sigma = u' = - \frac{\cos i}{p} \; \frac{\underline{r} \cdot \underline{v}}{r}$$

$$\rho = \sqrt{(\rho \sin \sigma)^2 + (\rho \cos \sigma)^2}$$

$$\sigma = \arctan \frac{\rho \sin \sigma}{\rho \cos \sigma}$$

5.5 In turn, longitude of node, right ascension and declination, and argument of latitude are found.

$$\sin \Omega = \frac{W_x}{\sin i} \; ; \; \cos \Omega = - \frac{W_y}{\sin i} \; ; \; \Omega = \arctan(\sin \Omega/\cos \Omega)$$

$$\sin \phi^* = \frac{Y}{\sqrt{X^2+Y^2}} ; \; \cos \phi^* = \frac{X}{\sqrt{X^2+Y^2}} \; ; \; \phi^* = \arctan \frac{\sin \phi}{\cos \phi}$$

$$\sin \theta^* = \frac{\sqrt{X^2+Y^2}}{r} \; ; \; \cos \theta^* = \frac{Z}{r} \; ; \; \theta^* = \arctan \frac{\sin \theta}{\cos \theta}$$

$$\cos \beta = \sin \theta^* [\cos \phi^* \cos \Omega + \sin \phi^* \sin \Omega]$$

$$\sin \beta = \frac{\cos \theta^*}{\sin i} \; ; \quad \beta = \arctan \frac{\sin \beta}{\cos \beta}$$

Conventional formulas are used to find the remaining osculating elements.

$$e \cos E = \frac{r\,v^2}{\mu} - 1\ ; \quad a = \frac{r}{1-e \cos E}$$

$$e \sin E = \frac{\underline{r} \cdot \underline{v}}{\sqrt{\mu a}}\ ; \quad e = \sqrt{(e \cos E)^2 + (e \sin E)^2}$$

$$\cos f = \frac{a}{r}\left(\frac{e \cos E}{e} - e\right)$$

$$\sin f = \frac{a}{r}\sqrt{1 - e^2}\left(\frac{e \sin E}{e}\right)$$

$$f = \arctan \frac{\sin f}{\cos f}$$

$$\text{Apogee height} = a(1+e) - R$$

$$\text{Perigee height} = a(1-e) - R$$

5.6 Other fundamental quantities computed from initial conditions are the constant introduced in Section 4

$$c_1 = -\frac{1}{2} A \cos^4 i_o \,(1-5 \cos^2 i_o)$$

the constant in the expansion of the energy integral

$$\bar{E} = \frac{\rho_o^2}{\cos^2 i_o} - 2\,\lambda u_o^3\left(\frac{1}{3} - \cos^2 \theta_0\right)$$

the energy itself

$$E = \frac{p^2}{2}\left[\frac{v_o^2}{p^2} - 2\lambda u_o^3\left(\frac{1}{3} - \cos^2 \theta_o\right) - 2Au_o\right]$$

and the period

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \quad .$$

5.7 The function B_1 of Section 4 is, in the present case, given by

$$B_1(i,\rho,\beta,\sigma) = QM\Theta_1 - QM\Theta_2 - Q\Theta_1 + Q\Theta_2$$

where

$$QM\Theta_1 = \pi \, A \cos^6 i$$

$$QM\Theta_2 = \frac{\pi}{2} A \cos^4 i \, (1 - 3 \cos^2 i)$$

$$Q\Theta_1 = \rho \cos^4 i \left[- \cos \sigma + \frac{1}{6} \cos (\sigma+2\beta) - \frac{1}{2} \cos (\sigma-2\beta) \right] + A \cos^6 i \left[\pi - \frac{1}{2} \sin 2\beta \right]$$

$$\begin{aligned} Q\Theta_2 = & - A \cos^4 i \sin^2 i \left[- \frac{3}{8} \sin (2\sigma+2\beta) - \frac{3}{2} \pi + \frac{3}{4} \sin 2\sigma + \frac{3}{4} \sin 2\beta \right] \\ & - \frac{A^2}{\rho} \cos^6 i \sin^2 i \left[- \frac{7}{12} \cos (\sigma+2\beta) - \frac{1}{4} \cos (2\beta-\sigma) + \frac{3}{2} \cos \sigma \right] \\ & - \rho \cos^2 i \sin^2 i \left[\frac{1}{16} \cos (3\sigma-2\beta) + \frac{7}{16} \cos (2\beta-\sigma) + \frac{1}{16} \cos (3\sigma+2\beta) - \frac{11}{48} \cos (\sigma+2\beta) + \frac{9}{8} \cos \sigma - \frac{1}{8} \cos 3\sigma \right] \end{aligned}$$

(formula continued next page)

$$-\rho \cos^2 i \left[-\frac{3}{4} \cos \sigma + \frac{1}{12} \cos 3\sigma \right]$$

$$-A \cos^4 i \left[\pi - \frac{1}{2} \sin 2\sigma \right]$$

$$+ \frac{A^2}{\rho} \cos^6 i \cos \sigma$$

The associated function b_1 is slightly less complicated

$$b_1(i,\rho,\beta,\sigma) = \frac{\sin (2\sigma-2\beta)}{2A \cos^4 i(1-5 \cos^2 i)} \; K$$

where

$$K = A^2 \cos^8 i \left[\sin^2 i \left(\frac{13}{4} \sin^2 i - \frac{13}{6}\right) + \cos^2 i \left(-\frac{5}{2} \sin^2 i + 2\right) \right]$$

$$+ \rho^2 \cos^4 i \left[\sin^2 i \left(\frac{65}{64} \sin^2 i - \frac{65}{96}\right) + \cos^2 i \left(-2 \sin^2 i + \frac{5}{6}\right) + \cos^4 i \right]$$

$$+ \frac{A^4}{\rho^2} \cos^{12} i \left[\sin^2 i \left(-\frac{3}{4} \sin^2 i + \frac{1}{2}\right) \right]$$

5.8 As usual, computation reverses the order of analysis, so the functions H_1 and h_1 of Section 3 are next to be evaluated.

$$H_1 = H_1(i,\rho,\beta,\sigma) = - QR_1 + h_1$$

where

$$QR_1 = \frac{1}{2} \rho \cos^3 i \sin i \left[\sin (\sigma-2\beta) + \frac{1}{3} \sin (\sigma+2\beta) \right]$$
$$+ \frac{1}{2} A \cos^5 i \sin i \cos 2\beta$$

$$h_1 = \frac{\rho^2 \sin 2 i}{2A(1-5\cos^2 i)} \left[\frac{\cos^2 i}{2} - \frac{\sin^2 i}{8} + \frac{1}{12} \right] \left[\cos (2\sigma-2\beta) + 1 \right]$$

Initially, the constant

$$p_1 = \lambda H_1(i_o,\rho_o,\beta_o,\sigma_o) + i_o$$

is also calculated.

5.9 After computing all the above quantities for the initial time t_o, the program is given $X_g, Y_g, Z_g, \dot{X}_g, \dot{Y}_g, \dot{Z}_g$ (output from the comparison method) at some later time t. Using this input in the formulas of 5.5 and 5.6, the program computes $r, v, u, i, \Omega, \phi^*, \theta^*, \beta, \sigma, \rho, f$, and apogee and perigee heights. In addition, the revolution number N is taken as the integral part of $(t-t_o)/P$

5.10 Using the value of σ just found, together with the various initial values, the program computes β by an iterative procedure. To start, we set

$$\beta_1 = \beta_o + (\sigma-\sigma_o)(1+\lambda c_1) + \lambda c_1(2\pi N)$$

Subsequently, we have

$$\begin{aligned}\beta_{k+1} = \beta_o &+ (\sigma-\sigma_o)(1+\lambda c_1) + \lambda c_1(2\pi N)\\ &+ \lambda\Big[b_1(i_o,\rho_o,\beta_o,\sigma_o) - b_1(i_o,\rho_o,\beta_k,\sigma)\\ &+ B_1(i_o,\rho_o,\beta_o,\sigma_o) - B_1(i_o,\rho_o,\beta_k,\sigma)\Big]\end{aligned}$$

and utilize the formulas of Section 5.7. The calculations are repeated as many times as are necessary to achieve convergence. In general, the value of β at the end of the third iteration was the same as that in the second. First order results, strictly speaking, can be obtained without iteration (here or below) but numerical results seemed improved by a few repetitions of the cycle.

5.11 With σ and β now known, another iterative proceedure is used to find i and ρ.

$$i_{k+1} = p_1 - \lambda H_1(i_k, \rho_k, \beta, \sigma)$$

$$\rho_{k+1}^2 = \cos^2 i_{k+1} \left\{ \bar{E} - A^2 \sin(i_o + i_{k+1}) \sin \left[\lambda H_1(i_o, \rho_o, \beta_o, \sigma_o) - \lambda H_1(i_{k+1}, \rho_k, \beta, \sigma) \right] + 2\lambda u_{k+1}^3 \left(\frac{1}{3} - \sin^2 i_{k+1} \sin^2 \beta\right) \right\}$$

where

$$u_{k+1} = \rho_k \sin \sigma + A \cos^2 i_{k+1}$$

This calculation starts with k = 0 and is repeated until the values of i and ρ converge. As a result of this computation $u(\beta)$ is known to first order. Note that we are here inverting the periodic integrals to find the periodic surface; see the end of Section 3.

5.12 Thus having obtained the values at time t of i, ρ, β, σ, and u, the program computes the corresponding position and velocity vectors, as follows:

$$\Omega = -\lambda \cos^3 i_o \left\{ A \cos^2 i_o \left[(\sigma-\sigma_o) + 2\pi N - \frac{1}{2} (\sin 2\beta - \sin 2\beta_o) \right] \right.$$

$$+ 2\rho_o \left[-\cos(\sigma_o-\beta_o) \left(\cos\beta - \frac{\cos^3\beta}{3} - \cos\beta_o + \frac{\cos^3\beta_o}{3} \right) \right.$$

$$\left. \left. + \frac{1}{3} \sin(\sigma_o-\beta_o) (\sin^3\beta - \sin^3\beta_o) \right] \right\}$$

$$+ \Omega_o$$

$$X_c = \frac{1}{u} \left[\cos\beta \cos\Omega - \sin\beta \cos i \sin\Omega \right]$$

$$Y_c = \frac{1}{u} \left[\sin\beta \cos i \cos\Omega + \cos\beta \sin\Omega \right]$$

$$Z_c = \frac{1}{u} \sin i \sin\beta$$

$$\Delta r = \sqrt{(X_c - X_g)^2 + (Y_c - Y_g)^2 + (Z_c - Z_g)^2}$$

$$\dot{\Omega} = -2\lambda p \cos^2 i \, u^3 \sin^2\beta$$

$$\dot{i} = -2\lambda p \cos^2 i \sin i (\sin\beta \cos\beta) u^3$$

$$\dot{\beta} = \frac{pu^2}{\cos i} \left[1 + 2\lambda u \cos^4 i \sin^2\beta \right]$$

$$\dot{r} = \frac{-p}{\cos i} (\rho \cos\sigma)$$

$$\dot{X} = \frac{1}{u} \left[-(\cos\beta \sin\Omega + \sin\beta \cos i \cos\Omega)\dot{\Omega} \right.$$

$$-(\sin\beta \cos\Omega + \cos\beta \cos i \sin\Omega)\dot{\beta}$$

$$\left. +(\sin\beta \sin i \sin\Omega)\dot{i} \right]$$

$$+ \dot{r} \left[\cos\beta \cos\Omega - \sin\beta \cos i \sin\Omega \right]$$

$$\dot{Y} = \frac{1}{u}\Big[-(\sin\beta\cos i\sin\Omega + \cos\beta\cos\Omega)\,\dot{\Omega}$$

$$+(\cos\beta\cos i\cos\Omega - \sin\beta\sin\Omega)\,\dot{\beta}$$

$$-(\sin\beta\sin i\cos\Omega)\,\dot{i}\Big]$$

$$+\,\dot{r}\Big[\sin\beta\cos i\cos\Omega + \cos\beta\sin\Omega\Big]$$

$$\dot{Z} = \frac{1}{u}\Big[\sin i\cos\beta\,\dot{\beta} + \cos i\sin\beta\,\dot{i}\Big]$$

$$+\,\dot{r}\Big[\sin i\sin\beta\Big]$$

$$\Delta V = \sqrt{(\dot{X}_c - \dot{X}_g)^2 + (\dot{Y}_c - \dot{Y}_g)^2 + (\dot{Z}_c - \dot{Z}_g)^2}$$

Finally the program computes the semi-major axis, eccentricity, apogee height, and perigee height.

$$e = \left[\frac{\rho^2}{A^2\cos^4 i}\right]^{1/2}$$

$$a = -\left\{2\left[\frac{E}{\mu} + \frac{JR^2}{r^3}\left(\frac{1}{3} - \sin^2 i\sin^2\beta\right)\right]\right\}^{-1}$$

Perigee height $= a(1-e) - R$

Apogee height $= a(1+e) - R$

Once these results have been printed, another set of comparison values may be read and computations, starting from 5.9, repeated.

The numerical results are given in the table below. The maximum Δr for each revolution is given in feet and is the square root of the sum of the squares of the differences between the Diliberto coordinates and those of a step by step integration run. The Diliberto coordinates were computed at 5 minute intervals during the first revolution and at twenty minute intervals thereafter. The comparison program utilizes the Cowell formulation with the Gauss-Jackson integration method and has been carefully tested to make certain that it is not, for present purposes, significantly in error. The initial values of inclination i and eccentricity e and the revolution number N are given. Runs were made for $e = 0.005$, 0.1, 0.2 and $i = 5^{\circ}$, $45^{\circ}, 63^{\circ}, 85^{\circ}$. It was obvious from values of Δr, even for the first revolution, that the Diliberto formulas given in this paper are not applicable to orbits of eccentricities as small as $e = 0.005$

	$i = 5^{\circ}$		$i = 45^{\circ}$	
N	e=0.1	e=0.2	e=0.1	e=0.2
1	2884	739	747	559
2	2953	883	960	451
3	3363	1112	1550	708
4	3605	1282	1720	882
5	3729	1390	2397	979
6	4006	1646	2504	1047
7	4088	1832	3245	1291
8	4413	1956	3288	1513
9	4455	2187	4085	1656
10	4820	2431	4063	1753
11	4812	2549	4919	1955
12	5212	2672	4946	2203
13	5170	2992	5766	2365
14	5609	3193	5829	2467
15	5532	3274	6584	2640
16	5995	3496	6707	2909
17	5886	3793	7391	3084
18	6389	3932	7582	3178
19	6343	3957	8214	3331
20	6787	4310	8453	3623

	$i = 63^\circ$		$i = 85^\circ$	
N	e=0.1	e=0.2	e=0.1	e=0.2
1	41793	4477	5557	986
2	38647	4321	7205	962
3	34372	4630	8889	1258
4	37268	4081	10566	1714
5	35749	3615	12413	1942
6	39620	4638	13876	1954
7	37045	4933	15942	2203
8	36104	4389	17230	2673
9	38402	3994	19459	2924
10	35516	4988	20593	2939
11	39683	5246	22972	3159
12	40986	4660	23921	3641
13	41005	4356	26490	3901
14	41410	5322	27306	3930
15	42282	5545	30045	4115
16	41798	4984	30651	4616
17	43579	4730	33572	4890
18	42217	5644	33994	4925
19	44873	5853	37079	5051

6. THE ANGLE-TIME RELATIONSHIP

The Diliberto expansion procedure has been used to determine the position and velocity of a satellite as a function of an inplane angle θ_2. Although these formulas are a first order approximation to the path of the satellite, the determination of position on the path at any instant requires knowledge of the relationship between θ_2 and the time, t. As in the simple case of elliptical motion, one obtains t as a function of θ_2, in effect, a generalization of Kepler's equation. It is necessary to invert by some approximate method, e.g. by iteration, in order to obtain $\theta_2(t)$.

The formulas which are discussed in this section have not yet been programmed and checked against a set of reference orbits. They were derived using the simpler of the two coordinates systems which have been studied, but the method can be used for the low eccentricity system.

We have as our definition of the variable w,

$$\frac{dw}{dt} = Hu^2 \tag{6.1}$$

Therefore,

$$\frac{dt}{d\theta_2} = \frac{dt}{dw}\,\frac{dw}{d\theta_2} = \sqrt{1 - r_1^2}\,\left[pu^2(1 + \lambda\theta_2)\right]^{-1} \tag{6.2}$$

The variables θ_1, r_1, r_2 which occur in the expression are known to first order as functions of θ_2. We have

$$r_k = p_k + \lambda S_k(q_1,q_2), \qquad k = 1,2,$$

$$\theta_1 = q_1 - \lambda \left[b_1(\Delta q) + B_1(q_1,q_2) \right], \tag{6.3}$$

$$\theta_2 = q_2, \; q_1 = q_1^o + (1 + \lambda c_1)q_2, \qquad \Delta q = q_2 - q_1 = - \lambda c_1 q_2 - q_1^o .$$

Let us write

$$\frac{dt}{d\theta_2} = V(q_1,q_2\lambda), \tag{6.4}$$

and decompose V into the sum of a function of Δq and a function with zero mean, i.e.

$$V(q_1,q_2,\lambda) = \ell(\Delta q,\lambda) + L(q_1,q_2,\lambda), \tag{6.5}$$

where

$$\ell(\Delta q,\lambda) = MV, \; L = V - MV ,$$

It will now be shown that the integral of V can be written as

$$\int_0^{q_2} V(q_1' , q_2',\lambda)\, dq_2' = -\frac{1}{c(\lambda)} \int_{-q_1^o}^{\Delta q} \ell(\psi)\, d\psi + QL + G(q_1,q_2,\lambda), \tag{6.6}$$

where the mean of the doubly periodic function G is a constant of the integration. The formulas in this representation of the integral of V have not been restricted to first order approximations. For example, we suppose $q_1 = q_1^o + (1 + c(\lambda))\, q_2$.

To determine the function G, we differentiate (6.6) and use the elementary fact that

$$\frac{d}{dq_2} QL(q_1,q_2,\lambda) = L + c(\lambda)\ Q \frac{\partial L}{\partial q_1} . \tag{6.7}$$

After cancellation, we obtain

$$- c(\lambda)\ Q \frac{\partial L}{\partial q_1} = (1 + c(\lambda)) \frac{\partial G}{\partial q_1} + \frac{\partial G}{\partial q_2} . \tag{6.8}$$

Up to this point, no approximation has been made. The existence and analyticity of the function G is known since V, ℓ, and L are analytic. The fact that G is doubly periodic follows from a special property of the function L, namely that it can be written as a finite Fourier series in q_1 with coefficients which are periodic functions of q_2. It follows that the integral of $L(q_1(q_2),q_2)$ is an almost periodic function with fundamental periods 2π and $2\pi/(1 + c)$. Since any such function can be written as a doubly periodic function in q_1 and q_2, the representation (6.6) is correct.

The explicit formula for G is found by expanding it in powers of λ and determining the coefficients from equation (6.8). Let

$$L = \sum_{n=0}^{\infty} \lambda^n L_n, \quad c(\lambda) = \sum_{n=1}^{\infty} \lambda^n c_n, \quad G = \sum_{n=0}^{\infty} \lambda^{n+1} G_n . \tag{6.9}$$

As usual, we obtain an infinite set of partial differential equations. No special analysis is needed to prove that periodic solutions can be obtained, since, by construction, the functions appearing on the right side have zero mean. The equations are

$$\frac{\partial G_o}{\partial q_1} + \frac{\partial G_o}{\partial q_2} = - c_1 \, Q \frac{\partial L_o}{\partial q_1} ,$$

$$\frac{\partial G_n}{\partial q_1} + \frac{\partial G_n}{\partial q_2} = - c_{n+1} Q \frac{\partial L_o}{\partial q_1} - \sum_{k=0}^{n-1} c_{k+1} \left[Q \frac{\partial L_{n-k}}{\partial q_1} + \frac{\partial G_{n-1+k}}{\partial q_1} \right] , \qquad n \geqq 1. \tag{6.10}$$

In the present problem,

$$V(q_1, q_2, \lambda) = V_o(q_2) + \lambda V_1(q_1, q_2) + O(\lambda^2). \tag{6.11}$$

Therefore $\partial L_o/\partial q_1 = 0$, and

$$t = \int_0^{q_2} V(q_1', q_2', \lambda) dq_2' = \int_0^{q_2} V_o(q_2') \, dq_2' - \frac{1}{c_1} \int_{-q_1}^{\Delta q} \int_o MV_1(\psi) d\psi + \lambda Q \left[V_1(q_1, q_2) - MV_1 \right]_0^{q_2} + O(\lambda^2) \tag{6.12}$$

with t measured from a time at which q_2 vanishes.

Explicit formulas for the functions appearing in this equation have been derived in terms of the variables used in the preceeding section. The development proceeds as follows. The function S_1 is already available, since to the required order

$$S_1(q_1,q_2) = - H_1(q_1,q_2,p_1,p_2)$$

and there is no difficulty in obtaining S_2 by substituting (6.3) in the energy integral and equating coefficients of λ. The expansion of V to get V_o and V_1 is straightforward and not too laborious; the first term in (6.11) is then readily identified with the true anomaly-time relationship for elliptic motion. At this point the direct approach seems to break down. While the necessary integrations can be carried through, terms proliferate until merely to set down the final expressions obtained for V_1,MV_1, and $Q(V_1-MV_1)$ would take some eleven pages here. Since these massive formulas are not particularly instructive, they are omitted.

7. FORMULAS - LOW ECCENTRICITY METHOD

The FORTRAN program for the low eccentricity method is a modification of the basic program and resembles it in many details, so we will concentrate here on the differences between the two programs and make frequent reference to Section 5.

7.1 The system (2.24) is fundamental. However, we set

$$\phi_1 = \beta, \; s_2 = \rho, \; \phi_2 = \sigma$$

and again make a change of the fourth variable so that i (not sin i) corresponds to s_1. We will not set forth here the complicated equations corresponding to those in 5.1, but be content to remark that θ_1 and the mean of θ_2 are as before. To be consistent with the notation of (2.24), we probably should write Φ_i instead of θ_i, but description of the low eccentricity program will be easier if we allow our symbols to duplicate those used for corresponding - though not necessarily identical - quantities in Section 5.

7.2 Calculations closely parallel those previously described. Input and evaluation of initial parameters are exactly as described in 5.4 and 5.5 except that we now have

$$\rho \sin \sigma = u - A \cos^2 i - \lambda \alpha_{11} - \lambda^2 \alpha_{12}$$

$$\rho \cos \sigma = u' - \lambda \alpha_{21} - \lambda^2 \alpha_{22}$$

where

$$\alpha_{11} = \frac{5}{6} A^2 \cos^6 i \sin^2 i \cos 2\beta + A^2 \cos^6 i \,(1-\frac{3}{2} \sin^2 i)$$

$$\alpha_{21} = \frac{1}{3} A^2 \sin^2 i \cos^6 i \sin 2\beta$$

$$\alpha_{12} = 2 A^3 \cos^{10} i \,(1- \frac{8}{3} \sin^2 i + \frac{17}{8} \sin^4 i)$$

$$+ \frac{1}{9} A^3 \sin^2 i \cos^{10} i \left[(44-61 \sin^2 i) \cos 2\beta + \frac{27}{4} \sin^2 i \cos 4\beta\right]$$

$$\alpha_{22} = - \frac{1}{9} A^3 \sin^2 i \cos^{10} i \left[2(2 - \sin^2 i) \sin 2\beta - 3 \sin^2 i \sin 4\beta\right]$$

and u′ is calculated as before.

7.3 Energy and the related constant $\bar{E}$ are found using variants of the previous formulas

$$\frac{2E}{p^2} = \frac{1}{\cos^2 i_o} \,(u_o'^2 + u_o^2) - 2u_o \left[A + \lambda u_o^2 \,(\frac{1}{3} - \cos^2 \theta_o)\right]$$

$$\bar{E} = \frac{2E}{p^2} + A^2 \cos^2 i_o$$

A change in the calculation of $\bar{E}$ is made necessary by the change in significance of ρ.

7.4 The functions B_1 and b_1 are evaluated as in 5.7. Since Θ_1 and M Θ_2 are identical with those for the basic method, it is only necessary to note that now

$$Q\Theta_2 = A\cos^4 i\,\sin^2 i\left\{\frac{3}{2}\pi - \frac{3}{4}\sin 2\beta - \frac{3}{4}\sin\sigma + \frac{3}{8}\sin(2\sigma + 2\beta)\right\}$$

$$+ \rho\sin^2 i\,\cos^2 i\left\{-\frac{9}{8}\cos\sigma - \frac{7}{16}\cos(2\beta-\sigma)\right.$$

$$+ \frac{11}{48}\cos(\sigma+2\beta) - \frac{1}{16}\cos(3\sigma-2\beta)$$

$$\left.+ \frac{1}{8}\cos(3\sigma) - \frac{1}{16}\cos(3\sigma+2\beta)\right\}$$

$$+ \rho\cos^2 r_1\left\{\frac{3}{4}\cos\sigma - \frac{1}{12}\cos 3\sigma.\right\}$$

$$+ A\cos^4 r_1\left\{-\pi + \frac{1}{2}\sin 2\sigma.\right\}$$

$$K = 2\left[\rho^2\cos^4 i\left(-\frac{3}{2}\cos^4 i - \frac{17}{12}\cos^2 i - \frac{65}{128}\sin^4 i - \frac{65}{192}\sin^2 i\right)\right.$$

$$+ A^2\cos^8 i\left(\frac{1}{4}\cos^2 i - \frac{3}{4}\cos^4 i\right.$$

$$\left.\left.+ \frac{3}{4}\sin^2 i - \frac{9}{8}\sin^4 i\right)\right]$$

There is no change in the treatment of H_1, p_1, and c_1.

75. After evaluating the necessary quantities at the initial time, the low eccentricity program, like the one from which it is derived, reads position and velocity components corresponding to some later time. Again the formulas of 5.5 and 5.6 (with low eccentricity modifications) are applied, as in 5.9, to obtain various parameters including a σ with which to enter the two iterative loops. The procedure for finding β is unchanged, but the iteration which yields u, ρ, and i is modified by using

$$u_{k+1} = \rho \sin \sigma + A \cos^2 i + \lambda\, \alpha_{11}(i_{k+1},\beta) + \lambda^2 \alpha_{21}(i_{k+1},\beta)$$

$$\begin{aligned}
\rho^2_{k+1} = \cos^2 i_{k+1} \Big[\bar{\bar{E}} + A^2 \sin (i_{k+1} + i_o) \sin (i_o - i_{k+1}) \Big] \\
- \lambda \Big[2\rho \cos \sigma\, \alpha_{21}(i_{k+1},\beta) + 2\rho \sin \sigma\, \alpha_{11}(i_{k+1},\beta) \\
- 2 \cos^2 i\, u^3_{k+1} \left(\tfrac{1}{3} - \sin^2 i_{k+1} \sin^2 \beta\right) \Big] \\
- \lambda^2 \Big[\alpha^2_{11}(i_{k+1},\beta) + \alpha^2_{21}(i_{k+1},\beta) \\
+ 2\rho \cos \sigma\, \alpha_{22}(i_{k+1},\beta) + 2\rho \sin \sigma\, \alpha_{12}(i_{k+1},\beta) \Big]
\end{aligned}$$

instead of the formulas which appear in 5.11.

7.6 Computation of position and velocity components, etc. is as described in 5.12, except that for calculating the eccentricity we use

$$\begin{aligned}
e^2 A^2 \cos^4 i = \rho^2 + \lambda \Big[2 \rho \cos \sigma\, \alpha_{21} + 2 \rho \sin \sigma\, \alpha_{11} \Big] \\
+ \lambda^2 \Big[\alpha^2_{11} + \alpha^2_{21} + 2 \rho \cos \sigma\, \alpha_{22} + 2 \rho \sin \sigma\, \alpha_{12} \Big]
\end{aligned}$$

Unfortunately, the low eccentricity modification has not given as satisfactory numerical results as the basic program. The β iteration converges, but to a limit which differs significantly from the value computed from the comparison coordinates. For one test orbit, there is a position error of ten thousand feet only five minutes from the epoch. Since the details of the analysis are much more complicated than the programming, which is of the sort in which

FORTRAN is particularly helpful, the difficulty probably lies in carrying out analytic proceedures according to the principles set forth in Sections 2, 3, and 4. To cope with such complicated routine analysis, we badly need more sophisticated programs for algebraic language manipulation.

8. POLAR ORBITS

In this section we apply periodic surface theory to the determination of polar orbits of a particle in the field of an oblate spheroid. There is some novelty in the treatment, as will be noted in the appropriate places. The orbit is plane; taking polar coordinates in this plane and writing

$$T = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2), \quad V = -\frac{\mu}{r} - \frac{\mu J R^2}{r^3} \left(\frac{1}{3} - \cos^2\theta\right),$$

we get the Lagrange equations

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} - \frac{3\mu J R^2}{r^4} \left(\frac{1}{3} - \cos^2\theta\right),$$

$$\frac{d}{dt} r^2\dot{\theta} = \frac{\mu J R^2}{r^3} \sin 2\theta.$$

Here R is the earth's equatorial radius, J is a dimensionless constant measuring the earth's oblateness, and $\mu = GM$, where G is the gravitational constant and M the earth's mass. The initial values at $t = 0$ are r_o, $\dot{r}_o$, θ_o, $\dot{\theta}_o$. We introduce

$$r^2\dot{\theta} = h, \quad u = \frac{1}{r}, \quad \lambda = \mu J R^2,$$

and take h and u as new dependent variables and θ as the new independent variable. Introducing $z = du/d\theta$ the equations for u,z, and h are

$$\frac{du}{d\theta} = z$$

$$\frac{dz}{d\theta} = -u + \frac{\mu}{h^2} + \lambda \left[\frac{3u^2}{h^2} \left(\frac{1}{3} - \cos^2\theta\right) - \frac{u}{h^2} \sin 2\theta \frac{du}{d\theta} \right], \tag{8.1}$$

$$\frac{dh}{d\theta} = \frac{\lambda\mu}{h} \sin 2\theta .$$

An integral of this system is the energy E, which has the expression, in terms of the variables u, z, h, and θ,

$$E = \frac{h^2}{2}\left[z^2 + u^2\right] - \mu u + \lambda u^3 \left(\frac{1}{6} + \frac{1}{2} \cos 2\theta\right). \tag{8.2}$$

In order to bring the equations (8.1) into a form suitable for the application of periodic surface theory we introduce the variables ρ and σ through the equations

$$u - \frac{\mu}{h^2} = \rho \sin \sigma , \quad \frac{du}{d\theta} = \rho \cos \sigma . \tag{8.3}$$

We get the equations

$$\frac{d\rho}{d\theta} = \lambda \left\{ \frac{2\mu}{h^4} u \sin \sigma \sin 2\theta - \cos \sigma \left[\frac{3}{h^2} u^2 \left(\frac{1}{6} + \frac{1}{2} \cos 2\theta\right) + \frac{\rho}{h^2} u \cos \sigma \sin 2\theta \right] \right\}$$

$$\frac{d\sigma}{d\theta} = 1 + \lambda \left\{ \frac{2\mu}{\rho h^4} u \cos \sigma \sin 2\theta + \frac{\sin \sigma}{\rho} \left[\frac{3}{h^2} u^2 \left(\frac{1}{6} + \frac{1}{2} \cos 2\theta\right) + \frac{\rho}{h^2} u \cos \sigma \sin 2\theta \right] \right\}$$

$$\frac{dh}{d\theta} = \frac{\lambda}{h} u \sin 2\theta , \quad \frac{d\theta}{d\theta} = 1 , \tag{8.4}$$

where u in equations (8.4) is obtained from (8.3). Let R_2, R_1, Θ_2, Θ_1 be the coefficient of λ in the equations for ρ, h, σ, and θ respectively ($\Theta_1 = 0$). Then a calculation shows

$$M(R_1) = 0, \quad M(\Theta_1) = 0, \quad M(R_2) = 0, \quad M(\Theta_2) = \frac{\mu}{2h^4} . \tag{8.5}$$

Since $M(R_1) = M(R_2) = 0$ we may calculate integrals of the form

$$h + \sum_{k=1}^{\infty} \lambda^k f_k(h, \rho, \sigma, \theta) = \text{const.}, \rho + \sum_{k=1}^{\infty} \lambda^k g_k (h, \rho, \sigma, \theta) = \text{const.},$$

using the Diliberto algorithm (described in Section 3). However we will consider an alternative change of variables; rather than ρ and σ we introduce E and σ, where E is given by (2) and σ by

$$\sigma = \tan^{-1} \left(u - \frac{\mu}{h^2}\right) \Big/ z . \tag{8.6}$$

We note that σ is the same in both variable changes. From (8.2) and (8.6) we get

$$\frac{1}{h^2}\left(2E + \frac{\mu^2}{h^2}\right) = \left(u - \frac{\mu}{h^2}\right)^2 \csc^2 \sigma + \frac{2\lambda u^3}{h^2}\left(\frac{1}{6} + \frac{1}{2}\cos 2\theta\right), \tag{8.7}$$

from which we get u as a function of E, h, σ and θ. We note that, to the first order in λ, we have

$$u = \frac{\mu}{h^2} + \zeta \sin \sigma - \frac{\lambda}{h^2 \zeta} \sin \sigma \left(\frac{\mu}{h^2} + \zeta \sin \sigma\right)^3\left(\frac{1}{6} + \frac{1}{2}\cos 2\theta\right),\ \zeta = \frac{1}{h}\left(2E + \frac{\mu^2}{h^2}\right)^{1/2}.$$

The equations for E, h, θ, and σ are (primes indicate differentiation with respect to θ)

$$E' = 0,\ h' = \frac{\lambda}{h}\left(\frac{\mu}{h^2} + \zeta \sin \sigma\right) \sin 2\theta,\ \theta' = 1, \tag{8.9}$$

$$\sigma' = 1 + \lambda\left\{\frac{2\mu}{h^4 \zeta}\left(\frac{\mu}{h^2} + \zeta \sin \sigma\right) \cos \sigma \sin 2\theta \right.$$

$$+ \frac{3 \sin \sigma}{h^2 \zeta}\left(\frac{\mu}{h^2} + \zeta \sin \sigma\right)^2\left(\frac{1}{6} + \frac{1}{2}\cos 2\theta\right)$$

$$\left. + \frac{1}{h^2} \sin \sigma \cos \sigma \sin 2\theta \left(\frac{\mu}{h^2} + \zeta \sin \sigma\right)\right\}.$$

The equations for h and σ are valid to the first order in λ since we have used only the 0^{th} order term in (8.8). Let the coefficients of λ in the equations for h and σ be R and Θ respectively.

For the system (8.9) we have the integral E = constant; we seek now, using the Diliberto algorithm, an integral of the form

$$h + \sum_{k=1}^{\infty} \lambda^k f_k(h, E, \sigma, \theta) = \alpha, \quad \alpha \text{ constant}, \tag{8.10}$$

where f_k has period 2π in σ and in θ. Using the operators M, Q, and P described in Section 3 a necessary condition for the existence of an integral of form (8.10) is MR = 0; this is easily verified. The function f_1 in (8.10) is given by

$$f_1(h, E, \sigma, \theta) = QR + g(h, E, \psi), \tag{8.11}$$

where $\psi = \sigma - \theta$ and g is the periodic solution (if it exists) of

$$\frac{\partial g}{\partial \psi} = \frac{MPQR}{M\Theta} \quad , \tag{8.12}$$

subject to some convenient normalization, e.g., $g(h,E,\frac{\pi}{2}) = 0$. A calculation shows that

$$QR = -\frac{\mu}{2h^3} \cos 2\theta + \frac{\zeta}{2h} \sin (2\theta-\sigma) - \frac{\zeta}{6h} \sin (2\theta+\sigma) , \tag{8.13}$$

$$M\Theta = \frac{\mu}{2h^4} ,$$

$$MPQR = \left(\frac{\mu^2}{8h^7} + \frac{\zeta^2}{12h^3}\right) \sin 2\psi - \frac{\zeta^2}{64h^3} \sin 4\psi .$$

Thus the periodic solution of (8.12) does exist and has the expression

$$g(h,E,\psi) = -\left(\frac{\mu}{3h^3} + \frac{\zeta^2 h}{12\mu}\right) (\cos 2\psi + 1) + \frac{\zeta^2 h}{128\mu} (\cos 4\psi - 1). \tag{8.14}$$

We now solve for h in $h + \lambda f_1(h,E,\sigma,\theta) = \alpha$; we get, valid to the first order in σ,

$$h = \alpha - \lambda f_1\ (\alpha,E,\sigma,\theta)\ , \tag{8.15}$$

and substituting (8.15) in the equation for σ in (8.9) we get again valid to be first order in λ,

$$\begin{aligned}\sigma' = 1 + \lambda \Bigg[& \frac{2\mu}{\alpha^4 \zeta_o} \left(\frac{\mu}{\alpha^2} + \zeta_o \sin \sigma\right) \cos \sigma \sin 2\theta \\ & + \frac{3 \sin \sigma}{\alpha^2 \zeta_o} \left(\frac{\mu}{\alpha^2} + \zeta_o \sin \sigma\right)^2 \left(\frac{1}{6} + \frac{1}{2} \cos 2\theta\right) \\ & + \frac{1}{\alpha^2} \sin \sigma \cos \sigma \sin 2\theta \left(\frac{\mu}{\alpha^2} + \zeta_o \sin \sigma\right)\Bigg]\ ,\end{aligned} \tag{8.16}$$

where $\zeta_o = (2E + \mu^2/\alpha)^{1/2}/\alpha$.

The solution of (8.16) can be obtained by Diliberto's second algorithm; however we will use here the simple procedure of replacing

σ on the right side of (8.16) by $\gamma + \theta$, where $\sigma = \sigma_o$ when $\theta = \theta_o$ and $\gamma = \sigma_o - \theta_o$. On making this substitution of $\gamma + \theta$ for σ let c be the constant term of the Fourier expansion of the coefficient of λ; the Fourier expansion has only a finite number of terms. A calculation shows that $c = \mu/2\alpha^4$. Then it is easily seen that the approximate solution of (8.16) is

$$\sigma = \gamma + (1+\lambda c)\theta + \lambda f(\theta), \tag{8.17}$$

where $f(\theta)$ is of period 2π and is given by

$$\begin{aligned} f(\theta) = - c\theta_o + \int_{\theta_o}^{\theta} \Big\{ - c &+ \frac{2\mu}{\alpha^4\zeta_o}\left(\frac{\mu}{\alpha^2} + \zeta_o \sin(\gamma+\theta)\right) \cos(\gamma+\theta) \sin 2\theta \\ &+ \frac{3}{\alpha^2\zeta_o} \sin(\gamma+\theta)\left(\frac{\mu}{\alpha^2} + \zeta_o \sin(\gamma+\theta)\right)^2\left(\frac{1}{6} + \frac{1}{2}\cos 2\theta\right) \\ &+ \frac{1}{\alpha^2} \sin(\gamma+\theta)\cos(\gamma+\theta)\sin 2\theta\left(\frac{\mu}{\alpha^2} + \zeta_o \sin(\gamma+\theta)\right) d\theta \\ = - \frac{\mu}{2\alpha^4}\theta_o + \int_{\theta_o}^{\theta} \Big[&\left(\frac{\mu^2}{2\alpha^6\zeta_o} + \frac{3\zeta_o}{8\alpha^2}\right)\sin(\theta+\gamma) + \left(\frac{\mu^2}{4\alpha^6\zeta_o} - \frac{11\,\zeta_o}{8\alpha^2}\right)\sin(\theta-\gamma) \\ &+ \frac{3\mu}{2\alpha^4}\cos 2\theta - \frac{\zeta_o}{16\,\alpha^2}\sin(\theta+2\gamma) - \frac{\mu}{2\alpha^4}\cos(2\theta+2\gamma) \\ &+ \left(\frac{7\mu^2}{4\alpha^6\zeta_o} + \frac{11\,\zeta_o}{16\,\alpha^2}\right)\sin(3\theta+\gamma) - \frac{\zeta_o}{8\alpha^2}\sin(3\theta+3\gamma) - \frac{3\mu}{2\alpha^4}\cos(4\theta+2\gamma) \\ &- \frac{5\zeta_o}{16\alpha^2}\sin(5\theta+3\gamma)\Big\}\, d\theta \,. \end{aligned} \tag{8.18}$$

We substitute the expression for σ (8.17) in (8.15) to obtain h as a function of θ. In doing this we make use of the approximation

$$\sin(2\theta-\sigma) = \sin(-\gamma + (1-\lambda c)\theta) - \lambda f(\theta)\cos(-\gamma+(1-\lambda c)\theta) \tag{8.19}$$

together with several other similar approximations. We get

$$\begin{aligned} h &= \alpha + \lambda\left[\frac{\mu}{2\alpha^3}\cos 2\theta - \frac{\zeta_o}{2\alpha}\sin(-\gamma + (1-\lambda c)\theta) + \frac{\zeta_o}{6\alpha}\sin(\gamma + (3+\lambda c)\theta)\right. \\ &\quad \left. + \left(\frac{\mu}{8\alpha^2} + \frac{\zeta_o^{\,2}\alpha}{12\mu}\right)(\cos 2(\gamma+\lambda c\theta) + 1) - \frac{\zeta_o^{\,2}\alpha}{128\mu}(\cos 4(\gamma+\lambda c\theta) - 1)\right] \\ &= \alpha + \lambda h_1(\theta)\,. \end{aligned} \tag{8.20}$$

This expression for h is valid to the first order in λ; we have neglected the second term on the right of (8.19), and the same term in the other similar approximations.

We are in position now to write $u = r^{-1}$ as a function of θ. Referring to (8.8) we will make the following substitutions in order to obtain a valid first order expression: in the coefficient of λ we will replace ζ by ζ_o, h by α, and σ by $\gamma + (1+\lambda c)\theta$; in the remaining terms we make the following substitutions

$$h = \alpha + \lambda h_1(\theta),$$

$$\zeta = \zeta_o + \frac{d\zeta_o}{d\alpha}(\lambda h_1(\theta)) = \zeta_o + \left(\frac{2E}{\zeta_o\alpha^3} - \frac{2\zeta_o}{\alpha}\right)\lambda h_1(\theta),$$

$$\sin\sigma = \sin(\gamma+(1+\lambda c)\theta) + \lambda f(\theta)\cos(\gamma+(1+\lambda c)\theta).$$

We get then the following expression

$$\frac{1}{r} = \frac{\mu}{\alpha^2} + \zeta_o \sin(\gamma+(1+\lambda c)\theta)$$

$$+ \lambda\left[\left[-\frac{2\mu}{\alpha^3} + \left(\frac{2E}{\zeta_o\alpha^3} - \frac{2\zeta_o}{\alpha}\right) \sin(\gamma+(1+\lambda c)\theta) \quad h_1(\theta)\right] + \zeta_o f(\theta) \cos(\gamma+(1+\lambda c)\theta)\right.$$

$$\left. - \frac{1}{\alpha^2\zeta_o} \sin(\gamma+(1+\lambda c)\theta) \left(\frac{\mu}{\alpha^2} + \zeta_o \sin(\gamma+(1+\lambda c)\theta)\right)^3\left(\frac{1}{6} + \frac{1}{2}\cos 2\theta\right)\right] \qquad (8.21)$$

It is clear that (21) is a valid approximation only if ζ_o is not small (high eccentricity case). If ζ_o is small it is necessary to apply the low eccentricity transformations described in Section 2.

9. EQUATORIAL ORBITS

If the earth gravitational potential is assumed to be independent of longitude, then a class of planar orbits is possible, namely, those in a plane containing the polar axis. Such orbits are discussed in Section 8. Another, and far simpler, class of planar orbits exists if the earth is assumed to be symmetric with respect to its equatorial plane. These orbits lie in the equatorial plane and (allowing only the second harmonic as a perturbation) are described by solutions of the differential equation

$$\frac{d^2u}{dw^2} + u = A + \lambda u^2 . \tag{9.1}$$

This special case has been used repeatedly in this report to illustrate and motivate the expansion techniques. At the expense of duplicating some of these statements, a more extensive analysis will be given in this section.

The specialization to equatorial orbits is frequently useful in the evaluation of a general perturbation scheme since it is easy to obtain both quantative and qualitative information about the solutions of the differential equation. These can then be compared with the formulas which are to be evaluated. Another reason for studying equatorial orbits is that some of the important properties of the general orbits can be examined in detail. For example, a satellite can be considered to be moving on a slowly rotating ellipse in its instantaneous orbit plane with a period (say, from perigee to perigee) depending on the oblateness parameter. This representation can be made precise for equatorial orbits. The formulas describing the orbits can be given exactly in terms of elliptic functions or approximated by truncating a convergent power series in the oblateness parameter. As was pointed out in Section 2, the small eccentricity difficulties can be recognized and easily overcome when only equatorial orbits are considered.

It should be emphasized that since equation (9.1) has periodic solutions corresponding to the unpertur ed elliptical orbits, the basis difficulty of the general problem disappears, namely that due to the (conjectured) almost periodic motion of the satellite. In particular, since the angular momentum vector is constant, the conjectured second integral becomes trivial, and the torus on which the orbits are assumed to lie becomes simply a closed curve. It should also be noted that while a periodic solution to the differential equation (9.1) determines a closed curve in the u, du/dw plane, the corresponding curve in the orbit plane need not be closed.

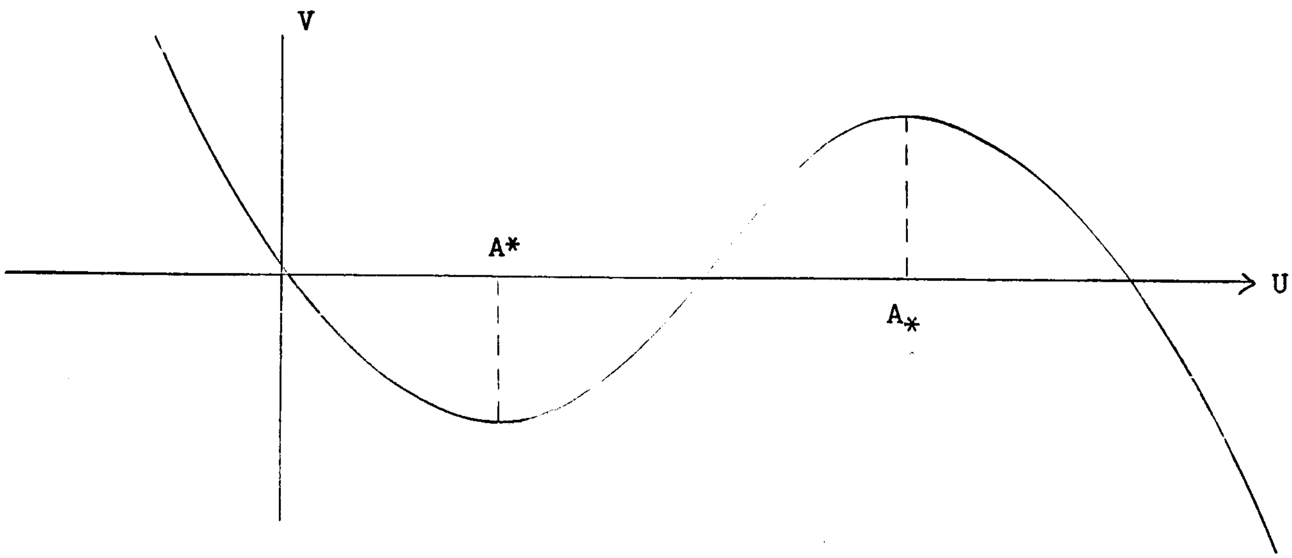
It is helpful to forget temporarily the source of equation (9.1) and to consider it as describing the one dimensional motion of a particle in a position dependent force field. The variable u can be considered as the displacement. The energy integral can be written

$$\left(\frac{du}{dw}\right)^2 + V(u) = 2EA/\mu, \qquad (9.2)$$

where the "potential energy" is given by

$$V(u) = u^2 - 2Au - \frac{2}{3}\lambda u^3 \qquad (9.3)$$

The basic qualitative properties of the solutions can be studied from the graph of V(u). We have



There are two equilibrium points A^* and A_*, which are the solutions to the quadratic equation

$$u = A + \lambda u^2 \qquad (9.4)$$

obtained from (9.1) by setting d^2u/dw^2 equal to zero. The circular orbit in the equatorial plane corresponds to the smaller root, A^*. If $0 < \lambda < 1/4$, the two roots are distinct. It is clear from the graph that the motion will be periodic if and only if

$$V(A^*) < 2EA/\mu < V(A_*) \; . \qquad (9.5)$$

As usual, we can find the period (in the w variable) by integrating

$$\frac{dw}{du} = [2EA/\mu - V(u)]^{-1/2} , \tag{9.6}$$

i.e.

$$\text{Period} = 2\pi/k(\lambda) = 2\int_{u_1}^{u_2} [2EA/\mu - V(u)]^{-1/2} du, \tag{9.7}$$

where $V(u_1) = V(u_2) = 2EA/\mu$, $u_1 < u_2$. Clearly $k(\lambda)$ is an analytic function with $k(0) = 1$. Since $V(u)$ is a cubic, equation (9.7) can be written in terms of elliptic integrals (see [7], Chapter XIII).

If the oblateness parameter is set equal to zero, the graph of $V(u)$ is a parabola and all the solutions for the differential equation are periodic. These solutions are classified by their energy

elliptic if $V(A) \leq 2EA/\mu < 0$

parabolic if $2EA/\mu = 0$

hyperbolic if $0 < 2EA/\mu$.

A similar classification can be made for the perturbed problem, however two new types of orbits must be considered. The first is a bounded (in u) non-periodic solution corresponding to the energy level $2EA/\mu = V(A_*)$. It is given by

$$u(w) = a + b \tanh^2 c(w - w_0), \tag{9.9}$$

where a, b and c can be found by substituting this expression into the differential equation (9.1). Orbits with higher energy are unbounded in u. This means that they spiral into the attracting body. For the perturbed problem, the orbits of primary interest are the perturbations of the elliptic orbits with energy bounded by $V(A^*) \leq 2EA/\mu < 0$. It should be noted that the non-zero angular momentum must be specified before the change in variable which gives equation (9.1) can be made.

Finally, we note that although the solution (9.9) is bounded in r if $0 < \lambda < 3/16$, a satellite on such an orbit would collide with the earth in very short time if the fact that the earth has finite extent is used.

Let us now return to the perturbation problem. Approximate solutions to (9.1) can be most easily generated by using the classical technique due to Lindstedt [5]. A new independent variable is introduced by $q = k(\lambda)w$, where $k(\lambda)$ is to be determined. (The notation of (9.7) is being used deliberately, since the two scale factors are in fact identical). Then with $u(q/k) = \bar{u}(q)$, we have

$$k(\lambda) \frac{d^2\bar{u}}{dq^2} + \bar{u} = A + \lambda \bar{u}^2 \tag{9.10}$$

Now we assume that

$$\bar{u}(q) = \sum_{j=0}^{\infty} \lambda^j \bar{u}_j(q) \tag{9.11}$$

where the $\bar{u}_j$ are periodic functions with period 2π. If we let

$$k^2(\lambda) = 1 + \sum_{j=0}^{\infty} \lambda^j a_j, \tag{9.12}$$

insert these expressions into (9.10) and equate coefficients of λ^j, we get an infinite set of differential equations of the form

$$\frac{d^2\bar{u}_j}{dq^2} + \bar{u}_j = L_j(q) - a_j \frac{d^2 u_o}{dq^2} , \tag{9.13}$$

where the L_j is a function of $\bar{u}_m$, $m < j$ and the constants a_m, $m < j$. For each such equation, the corresponding a_j can be uniquely chosen so that resonance is avoided, i.e., so that $\bar{u}_j(q)$ is a periodic function. This process can be justified by using the implicit function theorem to prove that the differential equation (9.1) has periodic solutions depending analytically on λ for appropriately restricted initial conditions. The period $2\pi/k(\lambda)$ is of course given by (9.7). For our present purposes the important fact is that the solution can be expanded in the form

$$u(w) = \sum_{j=0}^{\infty} \lambda^j \bar{u}_j(k(\lambda)w). \tag{9.14}$$

Let us now compare this with the formulas from the Diliberto procedure. As usual, we introduce polar coordinates

$$u = A^* + s \sin \phi, \quad \frac{du}{dw} = s \cos \phi . \tag{9.15}$$

The energy integral (2.2) becomes

$$s^2 = A^{*2} - \frac{4}{3}\lambda A^{*3} + 2\lambda A^* s^2 \sin^2 \phi + \frac{2}{3}\lambda s^3 \sin^3 \phi + \frac{2EA}{\mu} \quad (9.16)$$

This cubic equation can be solved for s as a function of ϕ.

We have

$$s = \sum_{j=0}^{\infty} \lambda^j s_j(\phi) \quad (9.17)$$

This is the equation of the periodic one-surface, i.e. the simple closed curve, on which the solution lies. The motion on this curve is described by the solution to (see(2.24))

$$\frac{d\phi}{dw} = 1 - \lambda \sin^2 \phi(2A^* + s(\phi) \sin \phi) \ . \quad (9.18)$$

A change in variable is now made,

$$q_1 = \phi + \sum_{n=1}^{\infty} \lambda^n B_n(\phi), \quad (9.19)$$

where the B_n are periodic functions with period 2π. They will be selected so that

$$\frac{dq_1}{dw} = 1 + \sum_{n=1}^{\infty} \lambda^n c_n \ . \quad (9.20)$$

Once this is done, (9.19) is inverted to give

$$\phi = q_1 + \sum_{n=1}^{\infty} \lambda^n A_n(q_1) \quad (9.21)$$

$$q_1 = (1 + \sum_{n=1}^{\infty} \lambda^n c_n)\, w$$

The solution u(w) is given by

$$u = A^* + s(\phi) \sin \phi \quad (9.22)$$

where $\phi(w)$ is defined by (9.21).

Comparing this to the Lindstedt formula (9.14), we see that

$$q = q_1 \text{ and } k(\lambda) = 1 + \sum_{n=1}^{\infty} \lambda^n c_n .$$

The Diliberto procedure is easily justified in this special case. We need only note that if $F(\phi,\lambda)$ is analytic in λ and has period 2π in ϕ. then if

$$\frac{d\phi}{dw} = 1 + \lambda F(\phi,\lambda), \tag{9.23}$$

we can determine $B(\phi,\lambda)$ and $k(\lambda)$ so that

$$q = \phi + \lambda B(\phi,\lambda) \text{ implies } \frac{dq}{dw} = k(\lambda) . \tag{9.24}$$

To do this, we differentiate the q equation and write the result as

$$\lambda \frac{dB}{d\phi} = - 1 + k(\lambda)/(1 + \lambda F) . \tag{9.25}$$

B will be periodic if $k(\lambda)$ is selected so that the mean of the right side of this equation vanishes. That is,

$$M [k(\lambda)/(1 + \lambda F)] = 1 . \tag{9.26}$$

Having made this choice, we integrate (9.25) to obtain the periodic function $B(\phi,\lambda)$. The variable q is now defined by (9.24).

By construction $dq/dw = k(\lambda)$. Since the functions are analytic in λ, the expansion procedure can be used to compute c_n and B_n.

The fact that the general procedure as given in Section 4 reduces to (9.19) can be shown by an induction argument which will not be given here. Finally, we note that if the small eccentricity change is not made, a similar argument can be given. However, the variable r_1 must be bounded away from zero, or the series will not converge.

10. AN ILLUSTRATIVE EXAMPLE

In the first sections of this report, the problem of recognizing the periodic surface is treated as follows: If the equations are given in rectangular coordinates we introduce polar coordinates in the phase space to bring the equations into the form (3.2). Periodic integrals of the form (3.3) are then sought; these integrals, solved for r_1 and r_2, are the periodic surfaces. The manner in which polar coordinates are introduced to bring the equations into form (3.2) is crucial. If it is not done properly integrals of the form (3.3) may not exist. It may be possible to change variables in equation (3.2) so that the new equations have the same form and, further, have periodic integrals with an expansion resembling (3.2); this is done in section 6 of [3]. This technique is not always applicable, as we will now show by presenting an example in which, although periodic surfaces do exist, polar coordinates must be introduced with care of integrals of the form (3.3) are to be obtained.

We consider the problem of finding periodic surfaces for the equations

$$\ddot{x} + x = \lambda y , \qquad \ddot{y} + y = \lambda x . \tag{10.1}$$

If we introduce the variables ξ and η

$$\xi = x + y , \quad \eta = x - y ,$$

we get the uncoupled equations

$$\ddot{\xi} + (1-\lambda)\, \xi = 0 , \quad \ddot{\eta} + (1+\lambda)\, \eta = 0 , \tag{10.2}$$

from which we get the integrals

$$\dot{\xi}^2 + (1-\lambda)\, \xi^2 = \text{const} , \quad \dot{\eta}^2 + (1+\lambda)\, \eta^2 = \text{const}, \tag{10.3}$$

or

$$(\dot{x}+\dot{y})^2 + (1-\lambda)(x+y)^2 = \text{const}, \quad (\dot{x}-\dot{y})^2 + (1+\lambda)(x-y)^2 = \text{const}. \tag{10.4}$$

These are the periodic surfaces in the phase space of $x, \dot{x}, y, \dot{y}$.
We can also see this if we write (10.2) in the following way

$$\frac{d\xi}{dt} = \dot{\xi} , \quad \frac{d\dot{\xi}}{dt} = - (1-\lambda)\, \xi , \quad \frac{d\eta}{dt} = \dot{\eta} , \quad \frac{d\dot{\eta}}{dt} = - (1+\lambda)\, \eta,$$

and introduce polar coordinates

$$\xi = r_1 \sin \theta_1 , \quad \dot{\xi} = r_1 \sqrt{1-\lambda} \cos \theta_1 \quad \eta = r_2 \sin \theta_2 , \ \dot{\eta} = r_2 \sqrt{1+\lambda} \cos \theta_2 .$$

We get then

$$\frac{dr_1}{dt} = 0 , \ \frac{d\theta_1}{dt} = \sqrt{1-\lambda} , \ \frac{dr_2}{dt} = 0 , \ \frac{d\theta_2}{dt} = \sqrt{1+\lambda}$$

so that r_1 = constant and r_2 = constant; this is equivalent to (10.3) and (10.4)

If however we write (10.1) in the form

$$\frac{dx}{dt} = \dot{x} , \quad \frac{d\dot{x}}{dt} = - x + \lambda y , \quad \frac{dy}{dt} = \dot{y} , \quad \frac{d\dot{y}}{dt} = - y + \lambda x$$

and introduce polar coordinates

$$x = r_1 \sin \theta_1 , \quad \dot{x} = r_1 \cos \theta_1 , \quad y = r_2 \sin \theta_2 , \quad \dot{y} = r_2 \cos \theta_2 \tag{10.5}$$

we get the equations

$$\begin{aligned}
\frac{dr_1}{dt} &= \lambda r_2 \cos \theta_1 \sin \theta_2 , \\
\frac{d\theta_1}{dt} &= 1 - \lambda \frac{r_2}{r_1} \sin \theta_1 \sin \theta_2 , \\
\frac{dr_2}{dt} &= \lambda r_1 \cos \theta_2 \sin \theta_1 , \\
\frac{d\theta_2}{dt} &= 1 - \lambda \frac{r_1}{r_2} \sin \theta_1 \sin \theta_2 .
\end{aligned} \tag{10.6}$$

The M operator applied to the right sides of the equations for r_1 and r_2 yields non-zero quantities. Thus there are no integrals of the form

$$r_1 + \sum \lambda^k f_k(r_1, r_2, \theta_1, \theta_2) = \text{const}, \quad r_2 + \sum \lambda^k g_k(r_1, r_2, \theta_1, \theta_2) = \text{const}. \tag{10.7}$$

On the other hand the periodic surfaces (10.4) do exist. Putting (10.5) in (10.4) we get the following form of the integrals of the system (10.6):

$$r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1) - \lambda (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2 = \text{const},$$

$$r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) + \lambda (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2 = \text{const}.$$

What this example indicates is that although periodic surfaces exist it may take more than the introduction of polar coordinates to detect them.

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